A Minimal Note on Weyl Group Representation Theory

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Theorem of the Highest Weight

Every irreducible representation contains a special vector that is highest with respect to the action of a certain Borel subalgebra (or the positive root space of a Cartan subalgebra). This means that a finite-dimensional irreducible representation is uniquely determined by its highest weight.

Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{h}$ its Cartan subalgebra. By choosing a set of positive roots Δ^+, any finite-dimensional representation V can be decomposed into a direct sum of weight spaces

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda, \quad V_\lambda = \{v \in V \mid h \cdot v = \lambda(h)v, \forall\, h \in \mathfrak{h}\}$$

In this decomposition, there exists a highest weight λ_0 that uniquely determines the structure of the representation V, satisfying

$$V_{\lambda_0} \neq \{0\} \quad \text{and} \quad V_{\lambda_0+\alpha} = \{0\} \quad \forall\, \alpha \in \Delta^+$$

Weyl Groups and Their Construction

A Weyl group is the finite Coxeter group generated by the reflections corresponding to the roots in a root system.

Definition

First, we define a root system and an inner product space.

- **Inner Product Space**: Let V be a real vector space equipped with a positive-definite inner product $(\cdot,\cdot)$. In many applications, V is taken to be the dual space of the Cartan subalgebra, $\mathfrak{h}^*$.
- **Root System**: Let $\Phi \subset V$ be a finite set satisfying:
  1. **Spanning**: Φ spans V.
  2. **Symmetry**: For any $\alpha \in \Phi$, the opposite $-\alpha$ is also in Φ.
  3. **Reflection Invariance**: For any $\alpha, \beta \in \Phi$, define the reflection

$$s_\alpha(\beta) = \beta - 2\frac{(\beta,\alpha)}{(\alpha,\alpha)}\alpha$$

Then $s_\alpha(\beta) \in \Phi$.

4. **Integrality Condition**: For all $\alpha, \beta \in \Phi$, we have $2\frac{(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$.

For each nonzero root $\alpha \in \Phi$, we define the corresponding reflection

$$s_\alpha : V \to V, \quad s_\alpha(v) = v - 2\frac{(v,\alpha)}{(\alpha,\alpha)}\alpha$$

Geometrically, this map represents a reflection about the hyperplane perpendicular to α and is an orthogonal transformation.

The group generated by all these reflections,

$$W = \langle s_\alpha \mid \alpha \in \Phi \rangle$$

is called the Weyl group of the root system. Since the root system is finite and each reflection is a linear transformation, the Weyl group is a finite reflection group.

Construction

Let $\mathfrak{g}$ be a semisimple Lie algebra and choose its Cartan subalgebra $\mathfrak{h}$. This gives rise to a root system $\Phi \subset \mathfrak{h}^*$, where each root α corresponds to a root space in $\mathfrak{g}$.

Within the root system, one selects a positive system (i.e., partitions Φ into positive and negative roots) and from the positive roots chooses a set of indecomposable minimal positive roots, called the set of **simple roots** Δ. Every root can be written as an integer linear combination of the simple roots, with nonnegative coefficients for positive roots.

For each simple root $\alpha \in \Delta$, construct the reflection s_α. The group is then generated by these reflections:

$$W = \langle s_\alpha \mid \alpha \in \Delta \rangle$$

Due to the symmetry of the root system, all reflections s_α (for $\alpha \in \Phi$) can be expressed in terms of the reflections corresponding to the simple roots; thus, this generated group already encompasses the entire Weyl group.

The Weyl group can be viewed as a Coxeter group with generators corresponding to the simple roots $\{s_\alpha \mid \alpha \in \Delta\}$, satisfying the relations

$$(s_\alpha s_\beta)^{m(\alpha,\beta)} = 1$$

where $m(\alpha, \beta)$ is an integer determined by the angle between the simple roots α and β:

- When $\alpha = \beta$, $m(\alpha, \alpha) = 1$ (since the square of a reflection is the identity).
- When the two roots form a 90° angle, $m(\alpha, \beta) = 2$; for a 120° angle, $m(\alpha, \beta) = 3$; for 135°, $m(\alpha, \beta) = 4$; and for 150°, $m(\alpha, \beta) = 6$.

Weyl Character Formula

I am not fond of the old-fashioned, enigmatic style of asking and answering one's own questions, but is there a way to compute the character of a finite-dimensional irreducible representation? Indeed, there is!

Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{h}$ its Cartan subalgebra, and let λ be a highest weight (dominant weight) such that $V(\lambda)$ is the irreducible representation with highest weight λ. Then its character is given by the expression:

$$\operatorname{ch} V(\lambda) = \frac{\sum_{w \in W} \epsilon(w) e^{w(\lambda+\rho)}}{\prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})}$$

where:

- W is the Weyl group (i.e., the Coxeter group generated by the root system),
- $\epsilon(w)$ is the determinant (or sign) of the Weyl group element w (namely, $+1$ or -1),
- Δ^+ is a set of positive roots, and
- ρ is the Weyl vector (half-sum of the positive roots), defined by $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$

Weyl Dimension Formula

In the character formula, one needs to set all formal exponentials to 1, but directly setting $e^\mu = 1$ leads to the indeterminate form $0/0$. To avoid this, we introduce a parameter t and replace the exponents in the formula with $e^{t\mu}$. That is, we consider

$$\operatorname{ch} V(\lambda; t) = \frac{\sum_{w \in W} \epsilon(w)\, e^{t\, w(\lambda+\rho)}}{\prod_{\alpha \in \Delta^+} \left(e^{t\alpha/2} - e^{-t\alpha/2} \right)}$$

For any μ, as $t \to 0$ we have

$$e^{t\mu} = 1 + t\mu + \frac{t^2 \mu^2}{2} + \cdots$$

For each factor in the denominator,

$$e^{t\alpha/2} - e^{-t\alpha/2} = 2 \sinh\left(\frac{t\alpha}{2} \right)$$

and as $t \to 0$, using the approximation $\sinh x \approx x$, we get

$$e^{t\alpha/2} - e^{-t\alpha/2} \approx t\alpha$$

Thus, the entire denominator is approximately

$$\prod_{\alpha \in \Delta^+} (e^{t\alpha/2} - e^{-t\alpha/2}) \approx t^{|\Delta^+|} \prod_{\alpha \in \Delta^+} \alpha$$

The numerator is

$$\sum_{w \in W} \epsilon(w)\, e^{t\, w(\lambda+\rho)}$$

which can be expanded as

$$\sum_{w \in W} \epsilon(w) \left(1 + t\, w(\lambda+\rho) + \frac{t^2}{2}\, w(\lambda+\rho)^2 + \cdots \right)$$

Notice that the constant term $\sum_{w \in W} \epsilon(w) = 0$ (when W is nontrivial), so the lowest nonzero term in the numerator is of order $O(t)$ or higher. Its order eventually cancels with the $t^{|\Delta^+|}$ in the denominator, yielding a finite limit.

Thus, as $t \to 0$, we can extract the dimension:

$$\operatorname{ch} V(\lambda;t)\Big|_{t\to 0} = \dim V(\lambda)$$

Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{h}$ its Cartan subalgebra, and Δ^+ the set of positive roots, with the Weyl vector defined as

$$\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$$

For an irreducible representation $V(\lambda)$ with highest weight λ, the Weyl dimension formula gives its dimension as

$$\dim V(\lambda) = \prod_{\alpha\in\Delta^+}\frac{\langle\lambda+\rho,\alpha\rangle}{\langle\rho,\alpha\rangle}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product on $\mathfrak{h}^*$.