Lower Semicontinuity and the Existence of Minimizers in Variational Problems

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In the calculus of variations, one of the most common problems can be abstractly formulated as

$$\inf_{u \in X} F(u),$$

where X is a function space (for example, a Banach space or a Hilbert space), and

$$F : X \to (-\infty, +\infty]$$

is a functional.

Problems of this form arise in many contexts, one of the most typical being the theory of weak solutions to partial differential equations.

This article is concerned with the **existence problem**. We do not discuss how to explicitly find a minimizer; instead, we focus on a single question: *Is the infimum actually attained by some* $u \in X$*?*

In other words, we want to know whether there exists $u^* \in X$ such that

$$F(u^*) = \inf_{u \in X} F(u).$$

It is important to note that the existence of a minimizer is logically independent of other properties. Even if a minimizer exists, it need not be unique; even if it is unique, it may fail to possess good regularity. Therefore, in the discussion that follows, we deliberately focus only on the question of existence, without addressing uniqueness or further qualitative properties of solutions.

In finite-dimensional spaces, such problems can often be handled directly using compactness and continuity. In infinite-dimensional spaces, however, the situation is fundamentally different. Typically, one cannot explicitly construct a minimizer, and instead adopts the following strategy: choose a sequence (u_n) such that the functional values approach the infimum. Such a sequence is called a minimizing sequence, and one then attempts to analyze its limiting behavior.

Consequently, the existence of minimizers in variational problems essentially depends on two key issues. First, whether such a sequence admits a sufficiently good limit; second, even if a limit exists, whether the functional value does not increase in the limiting process.

The Standard Structure of an Existence Proof

By the definition of the infimum, one can always choose a sequence $(u_n) \subset \mathcal{A}$ such that

$$F(u_n) \downarrow \inf_{u \in \mathcal{A}} F(u).$$

This sequence is called a minimizing sequence. This step is usually straightforward, as it merely consists of selecting points according to the definition of the infimum.

However, the fact that $F(u_n)$ approaches the minimum does not guarantee any form of convergence of u_n in the space X. A typical issue is that u_n may "escape to infinity" in X (that is, its norm diverges), while the values $F(u_n)$ still decrease.

Therefore, we need a mechanism that allows us to deduce boundedness of the sequence in X from boundedness of the energy. The most common condition used for this purpose is coercivity:

$$\|u\|_X \to \infty \quad \Rightarrow \quad F(u) \to \infty,$$

or, more commonly, an estimate of the form

$$F(u) \geq c\|u\|_X^p - C \quad (c > 0).$$

The role of this condition is straightforward. If $F(u_n)$ does not tend to infinity (as is the case for a minimizing sequence), then $\|u_n\|_X$ cannot diverge, and hence (u_n) is bounded in X.

In $\mathbb{R}^n$, every bounded sequence admits a strongly convergent subsequence. In infinite-dimensional spaces, however, boundedness alone generally does not imply the existence of a strongly convergent subsequence.

A common approach in the calculus of variations is therefore to work with weak convergence instead. In many spaces, a bounded sequence admits at least a weakly convergent subsequence. For example, if X is a reflexive Banach space (in particular, any Hilbert space), then

$$(u_n) \text{ bounded} \quad \Rightarrow \quad \exists \text{ a subsequence } u_{n_k} \rightharpoonup u.$$

The key idea here is that we accept a weaker notion of convergence in exchange for the existence of a limit object.

Even if $u_{n_k} \rightharpoonup u$, it does not automatically follow that $u \in \mathcal{A}$. If the constraint set is not preserved under taking limits, then the limit point cannot serve as a solution.

Therefore, we require that $\mathcal{A}$ be stable under weak convergence:

$$u_n \in \mathcal{A},\ u_n \rightharpoonup u \quad \Rightarrow \quad u \in \mathcal{A}.$$

In textbooks, this property is usually described by saying that $\mathcal{A}$ is weakly closed.

At this stage, we have obtained a candidate limit $u \in \mathcal{A}$. The remaining question is how to prove that it actually attains the minimum.

We know that

$$F(u_{n_k}) \to \inf_{\mathcal{A}} F,$$

but weak convergence is generally insufficient to imply $F(u_{n_k}) \to F(u)$. What we truly need is not equality, but an inequality in the correct direction:

$$F(u) \leq \liminf_{k\to\infty} F(u_{n_k}).$$

Once this inequality holds, we obtain

$$F(u) \leq \liminf F(u_{n_k}) = \inf_{\mathcal{A}} F,$$

and since $u \in \mathcal{A}$ implies $F(u) \geq \inf_{\mathcal{A}} F$, we conclude that

$$F(u) = \inf_{\mathcal{A}} F,$$

and hence a minimizer exists.

The appearance of the lim inf is natural: under weak convergence, functional values typically yield stable conclusions only in the "lower bound" direction.

Definition and Role of Lower Semicontinuity

Definition 0.1 (Weak Lower Semicontinuity)**.** A functional F is said to be weakly lower semicontinuous (w.l.s.c.) on X if for every sequence $u_n \rightharpoonup u$,

$$F(u) \le \liminf_{n\to\infty} F(u_n).$$

The role of this property can be summarized in one sentence: it guarantees that when we extract limits using weak convergence, the limit point is not worse than the sequence in the asymptotic sense, thereby allowing information about "approaching the minimum" to be transferred to the limit point.

We now state a directly applicable existence theorem (a standard template for the direct method).

Theorem 0.2 (Standard Version of the Direct Method)**.** *Let X be a reflexive Banach space and let $\mathcal{A} \subset X$ be nonempty and weakly closed. Assume that*

- $\inf_{\mathcal{A}} F > -\infty$*;*
- *F is coercive on $\mathcal{A}$ (or, more generally, coercive enough to ensure that every minimizing sequence is bounded in X);*
- *F is weakly lower semicontinuous on $\mathcal{A}$.*

Then there exists $u^{\in \mathcal{A}}$ such that

$$F(u^{)=\inf_{u\in\mathcal{A}} F(u).}$$

Remark. The proof of this theorem is essentially a rigorous formulation of the five steps outlined above. Many existence proofs in PDEs and variational problems amount to verifying these three conditions: boundedness, weak compactness, and lower semicontinuity.

Finally, we present a simple example illustrating that the existence of an infimum does not necessarily imply the existence of a minimizer.

Let $X = \mathbb{R}$ and define

$$F(x) = \begin{cases} |x|, & x \ne 0, \\ 1, & x = 0. \end{cases}$$

Clearly, $\inf_{\mathbb{R}} F = 0$, since choosing $x_n = \frac{1}{n}$ yields $F(x_n) = \frac{1}{n} \to 0$. However, there is no x such that $F(x) = 0$: if $x \ne 0$, then $F(x) = |x| > 0$; if $x = 0$, then $F(0) = 1$.

This phenomenon shows that a minimizing sequence can drive the functional value arbitrarily close to the infimum, while the limit point fails to inherit minimality. The root cause is that F is not lower semicontinuous at 0.