Complex analysis is not complex I-II

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Introduction

Complex Analysis is a branch of mathematical analysis that primarily studies the properties of complex functions, such as analyticity, integration, series, and their applications in fields like physics, engineering, and computer science. The structure of this article follows the typical syllabus of learning complex analysis, occasionally interspersed with personal insights and interpretations.

The content is tailored to readers familiar with mathematical concepts typically taught up to the middle school level.

1 Complex Numbers

1.1 Imaginary and Complex Numbers

The knowledge point most familiar to middle school students related to complex numbers often involves quadratic equations. When emphasizing that "the equation has no **real** roots," I always felt like adding more, but I restrained myself. A quadratic equation is typically written as:

$$ar^2 + br + c = 0$$

Previously, teachers would teach the "discriminant of a quadratic equation," denoted as $\Delta = b^2 - 4ac$, and then discuss it under the following three cases: + If $\Delta > 0$, the equation has two real roots: $r = \frac{-b \pm \sqrt{\Delta}}{2a}$ + If $\Delta = 0$, the equation has one real root: $r = \frac{-b}{2a}$ + If $\Delta < 0$, the equation has no real roots.

In the last case, since it is emphasized as "no real roots," it suggests there is something else. When I first learned this, the teacher smiled and casually asked, "Do you know what lies beyond real numbers?" Unfortunately, in eighth grade, my mathematical knowledge was limited to what was tested in exams, so I could only guess "imaginary numbers" as the opposite of "real." *Trying to sound smart but falling short*

We know that in the realm of real numbers, it is impossible to take the square root of a negative number. To extend the number system, people introduced $\sqrt{-1} = \pm i$, known as the "imaginary unit." However, it seems somewhat impractical—like with the discriminant—where $\Delta < 0$ geometrically implies no solutions. Thus, we generally think of imaginary numbers as an abstraction: "a number that lies somewhere between existence and non-existence." With i, the solutions to the equation when $\Delta < 0$ can take the form:

$$r_1 = \alpha + i\beta r_2 = \alpha - i\beta$$

At this point, the solutions are complex numbers, and they are conjugates of each other (to be explained shortly).

A complex number takes the form:

$$z = a + bi$$

Here, a and b are both real numbers. The term a is called the real part of the complex number, and b is called the imaginary part. Under this definition, the real part is often denoted as $\Re(z)$ or $\mathrm{Re}(z)$, and the imaginary part as $\Im(z)$ or $\mathrm{Im}(z)$. Thus:

$$z = \Re(z) + i\Im(z)$$

We can think of real numbers as complex numbers with an imaginary part equal to zero. Hence, the set of complex numbers $\mathbb{C}$ includes the set of real numbers $\mathbb{R}$.

1.1.1 Operations with Complex Numbers

To maintain compatibility with the original field of real numbers, we define operations for complex numbers. Consider two complex numbers $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$:

- $z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2)$
- $z_1 \cdot z_2 = (a_1 + b_1 i)(a_2 + b_2 i) = a_1 a_2 - b_1 b_2 + i(a_1 b_2 + b_1 a_2)$

Complex numbers satisfy the associative and commutative properties of addition and multiplication, as well as the distributive property of multiplication over addition. Additionally, complex numbers have additive and multiplicative identities, as well as their respective inverses.

Unlike real numbers, complex numbers cannot be ordered in a way that satisfies a total order. To establish a total order for complex numbers, the following conditions must hold for any $z_1, z_2, z_3 \in \mathbb{C}$:

1. If $z_1 \leq z_2$, then $z_1 + z_3 \leq z_2 + z_3$.
2. If $z_1 \leq z_2$ and $z_3 > 0$, then $z_1 z_3 \leq z_2 z_3$.

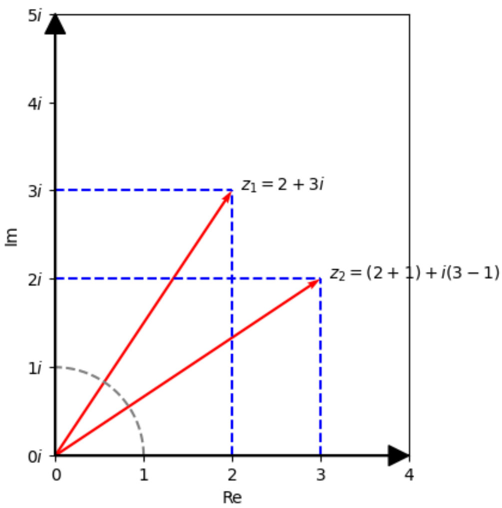
Let us consider the imaginary unit i as an example. For a total order to exist, one of the following must hold: $i > 0$, $i < 0$, or $i = 0$. Since $i \neq 0$, we examine the possibilities:

- If $i > 0$, by condition (2), $i^2 > 0$ must hold. However, $i^2 = -1$, and $-1 > 0$ is false.
- If $i < 0$, then $-i > 0$. By condition (2), $(-i)^2 > 0$ must hold. However, $(-i)^2 = -1$, and $-1 > 0$ is also false.

Thus, it is impossible to define a total order for complex numbers as we do for real numbers.

1.2 Complex Plane

To explore the geometric representation and significance of complex numbers, we need to establish a suitable plane.



Any complex number $z = a + bi$ can be uniquely represented by an ordered pair (a, b). Thus, we can establish a one-to-one correspondence between the set of complex numbers and points in a Cartesian coordinate plane. The collection of these points is called the plane point set, or simply the point set. Based on this, we generally refer to the horizontal axis as the real axis and the vertical axis as the imaginary axis. It is evident that, except for the origin, points on the imaginary axis represent imaginary numbers. Hence, our representation $z(a, b)$ is equivalent to $z = a + bi$.

Moreover, representing complex numbers as vectors can greatly aid in understanding complex number operations (as shown in the figure). For instance, the addition of complex numbers can be interpreted using the parallelogram rule for vectors. The length of vector $\mathbf{z}$ is called the modulus of the complex number z, defined as:

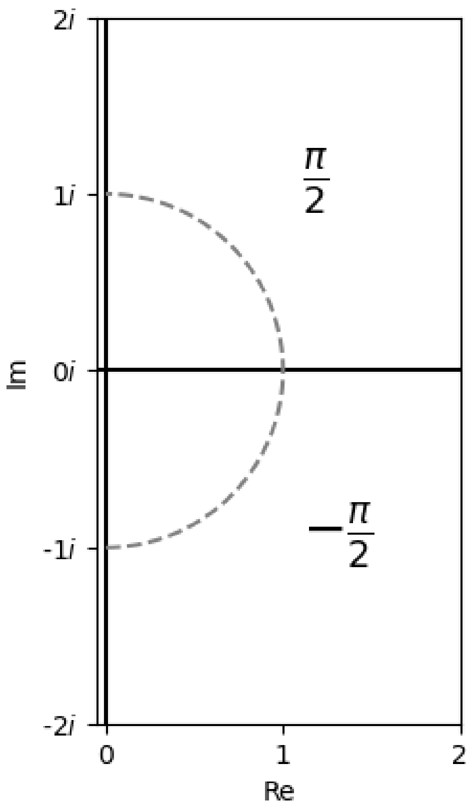
$$r = |\mathbf{z}| = \sqrt{a^2 + b^2} > 0$$

- $|z| - |w| \leq |z + w| \leq |z| + |w|$
- $|zw| = |z||w|$
- $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$

Additionally, complex numbers can also be represented in polar coordinates. If the angle between the vector and the real axis is denoted as $\theta = \arg z$, it is called the argument of the complex number z. By convention, the positive direction of the angle is counterclockwise. In the figure, the argument of i is $\frac{\pi}{2}$, while the argument of $-i$ is the opposite.

Every nonzero complex number has infinitely many arguments, differing by integer multiples of 2π. These values are defined as $\{\theta + 2k\pi \mid k \in \mathbb{Z}\}$. We often restrict θ to satisfy $-\pi < \theta \leq \pi$ as the principal value of the argument (sometimes $0 < \theta \leq 2\pi$ is used instead, depending on the convention). Since:

$$\begin{cases} a = r\cos\theta \\ b = r\sin\theta \end{cases}$$



We can express the complex number as $z = a + bi = r(\cos\theta + i\sin\theta)$. This is called the trigonometric form or polar form of a complex number. Sometimes, $\cos\theta + i\sin\theta$ is abbreviated as cis θ. (However, since LaTeX does not have a default formula for cis, this notation can be inconvenient, and I may avoid using it in the future.)

1.2.1 Euler's Formula

One of the most important formulas in mathematics and physics is Euler's formula, often referred to as the "God's formula":

$$e^{ix} = \cos x + i\sin x$$

Many books and documents describe a "derivation" of Euler's formula, such as expanding both sides using Taylor series for the exponential and trigonometric functions. Strictly speaking, such methods are validations rather than true derivations. Due to space constraints, I won't delve too deeply, but here are some superficial insights I have about Euler's formula.

Let us consider the unit circle, which is a circle centered at the origin with a radius of 1. In this case, the polar form of a complex number (with an angle x relative to the real axis) becomes:

$$y = \cos\theta + i\sin\theta$$

Note that multiplying a complex number by i corresponds to rotating the vector counterclockwise by 90 degrees in the complex plane. This rotational interpretation of i is closely related to its meaning. Euler's formula implies that the trajectory traced by the complex number e^{ix} on the complex plane is precisely the points on the unit circle. Thus, we are essentially concerned with a complex number with a modulus of 1.

During this rotation, the modulus of the vector remains unchanged, but the angle x and the vector y change. A functional relationship begins to emerge as we observe how y changes with the angle. At any point, the real part of y is $\cos x$, and the imaginary part is $i\sin x$. As y continues to rotate, the infinitesimal change in its rotation (the derivative) corresponds to the change in the complex number, which we denote as $y' = \frac{\mathrm{d}y}{\mathrm{d}x}$. This change effectively represents a counterclockwise rotation by 90 degrees, which is equivalent to multiplying by i. (We can easily visualize the original y and the rotated y', and consider the vector $\overline{y}$ representing their difference, which is perpendicular to y.)

This leads us to the following differential equation. Substituting $(0, 1)$ into it (details omitted for brevity, as a few simple steps yield the result), we get:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = iy = i(\cos\theta + i\sin\theta)$$

1.2.2 Conjugate

The term "conjugate" comes from the ancient concept of coupling two oxen side by side to pull a cart, with the "yoke" representing the idea of things being paired together, but possibly opposite in certain properties. In mathematics, we have the concept of complex conjugates and conjugacy classes (related to Group Theory), and in chemistry, there are conjugate acid-base pairs. The fundamental idea is similar: things that are "coupled" but have opposite properties in certain aspects.

In the study of complex numbers, the conjugate refers to the complex number obtained by changing the sign of the imaginary part. A pair of complex numbers with the same real part and opposite imaginary parts is called "conjugate complex numbers."

Consider a complex number $z = a + bi$, then the conjugate of z is:

$$\overline{z} = \overline{a + bi} = a - bi$$

It's easy to imagine that, when represented as vectors in the complex plane, they are symmetric with respect to the real axis—this also implies that the conjugate of a real number is the number itself.

Here are some basic properties of complex conjugates, which can be easily derived and apply to many situations:

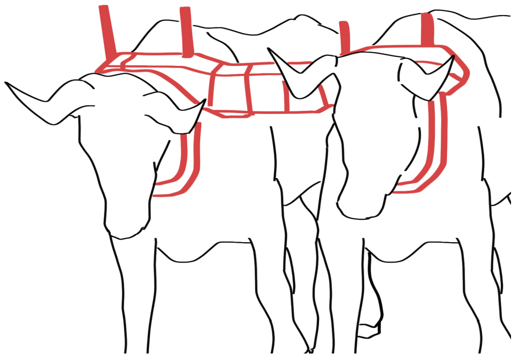


Figure 1: I prefer not to use real object images that disrupt the overall style, so here's my attempt at a hand-drawn interpretation (I did my best, haha)

- $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
- $\overline{z_1 \times z_2} = \overline{z_1} \times \overline{z_2}$
- $|z| = |\overline{z}|$
- For any integer $n \in \mathbb{Z}$, $\overline{z^n} = \overline{z}^n$

1.3 Riemann Sphere

The Riemann sphere can be considered as an extension of the complex plane. Each point on the complex plane is mapped to the unit sphere in three-dimensional space, with the point at infinity on the complex plane being mapped to the north pole of the sphere. The resulting sphere is known as the Riemann sphere (or the complex sphere), which is essentially a more geometric version of $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Compactifying the complex plane is important for our subsequent study of its properties.

When we establish a Cartesian coordinate system for the Riemann sphere, with the center of the sphere at the origin O of the complex plane, for every point on the unit sphere in three-dimensional space, we have the equation $X^2 + Y^2 + Z^2 = 1$, with the north pole located at $(0, 0, 1)$. It is important to note that, in contrast to the extended real axis $\overline{\mathbb{R}}$ which has two points at infinity, the extended complex plane only has one point at infinity, the north pole.

In our understanding of the real number line and the ordinary complex plane, division by zero is undefined. However, on the Riemann sphere, this can be defined. Division by zero is not defined on the extended real axis because positive infinity and negative infinity are on opposite sides of the number line, and we cannot treat them as the same point or "constrict" them together. But on the Riemann sphere, geometrically, it's easy to imagine that as a point on the sphere moves farther from the plane, its projection approaches the north pole. As a point on the complex plane moves toward infinity, its projection on the sphere tends to the north pole. Thus, dividing any non-zero number by zero can be understood as that number approaching infinity, so in the limit, any non-zero number divided by zero is infinite.

1.3.1 Algebraic Geometry Perspective

Additionally, it's worth briefly mentioning the role of the Riemann sphere in other branches of mathematics. The complex projective space refers to the complex plane plus a point at infinity, forming a topological space. The Riemann sphere is a compact one-dimensional complex projective space (also referred to as the complex projective line), denoted by $\mathbb{CP}^1$. *As is well-known, when we describe something as "compact," we are emphasizing that it is a well-behaved object, and mathematics is no exception.* Moreover, the Riemann sphere is a typical Riemann surface and a compact one-dimensional complex surface with many wonderful topological properties.

A complex algebraic variety can be defined by a set of polynomial equations. In the case of one-dimensional complex projective space, we can define it using the equation of the sphere. A point (x, y, z) on the sphere satisfies the equation $x^2 + y^2 + z^2 = 1$, meaning that the Riemann sphere is a one-dimensional complex algebraic variety.

In algebraic geometry, projective geometry, and other fields, the Riemann sphere frequently appears as a typical and intuitive example for various concepts, which is very helpful for our further study.

2 Complex Analytic Functions - Introduction

The concept of analytic functions can be further divided into real analytic functions (which are generally referred to as "infinitely differentiable" rather than analytic) and complex analytic functions. The latter is the main focus of our study in complex analysis, as they have many important properties.

In the first chapter, we gained a basic understanding of complex numbers and the complex plane. Now, our exciting journey into complex analysis begins!

2.1 Complex Functions and Their Limits

A complex function (also called a complex-variable function, which are equivalent terms, and I will use whichever feels more natural) is a function where both the independent and dependent variables are complex numbers. The domain and codomain of such functions are subsets of the complex plane. Let's define this formally:

Consider $E \subseteq \mathbb{C}$. We define a mapping $f : E \to \mathbb{C}$ as a complex function on E. It can be written as two real-valued functions in the following form:

$$f(z) = u(x, y) + iv(x, y)$$

To study the properties of complex functions, we often convert them into real functions. For example, we can transform the function $f(z) = z^2 + 1$ into a pair of real functions:

$$z = x + yif(z) = z^2 + 1 = (x^2 + y^2 + 1) + i2xy$$

Thus, we have $u = x^2 + y^2 + 1$ and $v = 2xy$.

If for all $z \in E$ there is a uniquely determined complex number $w = f(z)$, we say that w is a single-valued function on E, denoted as $w = f(z)$ $(z \in E)$. Otherwise, we call w a multi-valued function on E, such as for power and logarithmic functions.

The definition of the limit for real-valued functions is similar to that for simple one-variable real functions. Consider a function $w = f(z)$ defined on E. For any $\varepsilon > 0$, there exists a $\delta > 0$ such that when $|z - z_0| < \delta$, we have $|f(z) - L| < \varepsilon$. This means we say the function f has a limit L as z approaches z_0, denoted:

$$\lim_{z \to z_0} f(z) = L$$

Since we know that a complex function can be expressed as two real functions, we can convert the limit problem for complex functions into a limit problem for real-valued functions. The limit of z approaching a complex number $z_0 = x_0 + iy_0$ exists if and only if both of the real functions $u(x, y)$ and $v(x, y)$ are continuous at (x_0, y_0).

2.1.1 Complex Differentiability and Analytic Functions

If a function $w = f(z)$ is defined in a neighborhood of a point $z_0 \in E$ and has a finite limit:

$$\lim_{\Delta z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

we say that $f(z)$ is differentiable at z_0. This limit is also called the derivative, and we denote it as:

$$\Delta z = z - z_0 f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{z - z_0}$$

Clearly, if $f(z)$ is differentiable at z_0, then $f(z)$ is continuous at z_0.

When a function is differentiable at z_0 and in its neighborhood $B(z_0, \delta)$, we say that $f(z)$ is analytic at z_0 (however, note that "analytic" refers to the property of a function over a region, not just at a specific point). The domain of analyticity can also refer to a larger region, like an interval, meaning that it is a function that is differentiable everywhere within that region. If a function is analytic at a point, it must be differentiable at that point, but the reverse is not necessarily true.

Expanding this concept to the domain, for a complex function $f(z)$, if at every point z_0 in the domain $E \subset \mathbb{C}$ and in its neighborhood there exists a continuous derivative $f'(z_0)$, then we say that $f(z)$ is an analytic function (also called **holomorphic function**) on E. This property means that the function is differentiable at every point in the domain.

2.2 Cauchy-Riemann Equations

One of the most fundamental and important formulas in complex analysis is the Cauchy-Riemann equations (CRE), which provide a necessary and sufficient condition for a differentiable function to be analytic in its domain.

Consider a function $f(z) = u(x, y) + iv(x, y)$ in a domain E and a complex number $z = x + iy$. The Cauchy-Riemann equations are:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

We say that the function f is differentiable at a point z in E if and only if it satisfies the Cauchy-Riemann equations at z. Furthermore, f is analytic on E if and only if the Cauchy-Riemann equations are satisfied.

Often, to make boundary value problems in circular or annular regions easier to handle, we consider the Cauchy-Riemann equations in polar coordinates. For a complex number $z = re^{i\theta}$, in polar coordinates $f(z) = u(r, \theta) + iv(r, \theta)$, the Cauchy-Riemann equations can also be expressed as:

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \ \frac{\partial u}{\partial \theta} = -\frac{1}{r}\frac{\partial v}{\partial r}$$

The Cauchy-Riemann equations also imply an important point: the real and imaginary parts of a differentiable complex function are not independent of each other; they are related.

3 Series Theory

The theory of series for complex functions is one of the most fundamental and important parts of complex analysis. It will play a crucial role in the study of analytic functions, meromorphic functions, and Riemann surfaces, among other topics.

3.1 Complex Series

The definition and properties of complex series are similar to those of real series. Given a complex sequence $\{z_n\}_{n=1}^{\infty}$, a complex series is of the form:

$$\sum_{n=1}^{\infty} z_n = z_1 + z_2 + z_3 \ldots + z_n + \ldots$$

The sum of the first n terms of the complex sequence, S_n, is called the partial sum of the series.

Similarly, complex series have convergence and divergence. If the limit of the sequence of partial sums does not exist, we say that the series $\sum_{n=1}^{\infty} z_n$ diverges. Let $z_n = x_n + iy_n, \ x_n, y_n \in \mathbb{R}$. We define the real part series $\sum_{n=1}^{\infty} x_n$ and the imaginary part series $\sum_{n=1}^{\infty} y_n$. If the real part series and the imaginary part series converge to some x and y respectively, then the complex series $\sum_{n=1}^{\infty} z_n$ converges to $x + iy$.

If $\sum_{n=1}^{\infty} |z_n|$ converges, we say that $\sum_{n=1}^{\infty} z_n$ converges absolutely.

Complex series satisfy the property of linear combinations, meaning if the series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ both converge, then $\sum_{n=0}^{\infty}(ka_n + b_n)$ also converges, where k is any complex number.

The convergence and properties of complex series are similar to those of real series, and we can determine whether a series converges using convergence tests (also called convergence criteria), such as the comparison test, ratio test, and root test. Continuing from the previous definition, here are some of the most basic tests. Consider two series $\sum_{n=0}^{\infty} u_n$ and $\sum_{n=0}^{\infty} v_n$, where we are determining the convergence of $\sum_{n=0}^{\infty} u_n$.

- **Comparison Test**: If $|u_n| \leq v_n$ and $\sum_{n=0}^{\infty} v_n$ converges, then $\sum_{n=0}^{\infty} u_n$ also converges.
- **Ratio Test (D'Alembert's Criterion)**: Let $\rho = \lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right|$. If:
  - $r < 1$, the series converges absolutely.
  - $r > 1$ or $r = \infty$, the series diverges.
  - $r = 1$, the series may converge or diverge, and further tests are needed.
- **Root Test (Cauchy's Criterion)**: Let the root value of the series be $r = \limsup_{n\to\infty} \sqrt[n]{|u_n|}$. If:
  - $r < 1$, the series converges absolutely.
  - $r > 1$ or $r = \infty$, the series diverges.
  - $r = 1$, the series may converge or diverge, and further tests are needed.

3.2 Complex Function Series

Similarly, complex function series are analogous to real function series. Given a sequence of complex functions $\{f(z)\}_{n=1}^{\infty}$ with disjoint domains, we define their intersection as D, which is the domain of the series. A complex function series is of the form:

$$\sum_{n=1}^{\infty} f_n(z) = f_1(z) + f_2(z) + f_3(z) + \ldots + f_n(z) + \ldots$$

The sum of the first n terms of the complex function sequence, S_n, is called the partial sum of the series.

For a complex function series $\sum_{n=1}^{\infty} f_n(z)$, for each point z in the domain D, for any $\varepsilon > 0$, there exists an N_z such that for $n > N_z$, for any positive integer p, we have:

$$\left|\sum_{i=n+1}^{n+p} f_i(z)\right| < \varepsilon$$

We say that the complex function series is convergent. Furthermore, if there exists a natural number N (independent of z) that satisfies the condition for every z in D, namely when $n > N$, for every positive integer p, we have $\left|\sum_{i=n+1}^{n+p} f_i(z)\right| < \varepsilon$, then the series is uniformly convergent.

The definition may seem complicated, but the idea is simple: at each point z, the partial sums of the series get arbitrarily close to zero, and this property holds uniformly for all z in the domain D.

3.2.1 Weierstrass M-test

For function series, there is a method similar to the comparison test for convergence, called the Weierstrass M-test (Weierstrasssches Majorantenkriterium, to experience the charm of the German language), often abbreviated as the "M-test" in Chinese.

Consider a sequence of functions $\{f_n(z)\}$ and its domain D, along with a constant M_n such that:

$$|f_n(z)| \leq M_n$$

for all $n \geq 1$ and for all z in D. If the series $\sum_{n=1}^{\infty} M_n$ converges, then the series $\sum_{n=1}^{\infty} f_n(z)$ converges uniformly on D.

The core idea of the Weierstrass M-test is that, as the sequence of functions $\{f_n\}$ converges to a limit function $f(z)$ as n increases, we require that the function sequence at each point z be bounded by the same upper bound M, meaning the absolute values of the functions are controlled by a constant. This allows us to determine uniform convergence.

3.2.2 Complex Power Series

A common special type of complex function series is called a complex power series. Because it has distinct convergence criteria and better properties compared to general complex function series, it is worth discussing separately.

Consider complex numbers z_0 and c_n, where a complex power series takes the form:

$$\sum_{n=1}^{\infty} c_n(z - z_0)^n$$

This is called a complex power series centered at z_0. For the series $\sum_{n=1}^{\infty} c_n(z - z_0)^n$, the radius of convergence refers to the radius within which the power series $\sum_{n=0}^{\infty} c_n z^n$ converges in the complex plane. If there exists $R > 0$ such that the series converges absolutely within the disk $|z| < R$ centered at z_0, and diverges on the boundary $|z| = R$, then R is called the radius of convergence of the series. The region where the complex power series converges is called the convergence disk, and within the convergence disk, the power series represents an analytic function.

Specifically, when $z_0 = 0$, i.e., the series is centered at 0, the complex power series takes the form:

$$\sum_{n=1}^{\infty} c_n z^n$$

This is called the Maclaurin series.

4 Complex Integration

For convenience, we make the following assumptions about the orientation of curves in this article, unless otherwise specified:

- All curves are smooth or piecewise smooth, meaning they are either piecewise differentiable or have a well-defined length.
- For an open arc, we typically specify the starting and ending points.

- For closed curves, counterclockwise is considered the positive direction, and is denoted as C^+; clockwise is considered the negative direction, denoted as C^-.

4.1 Concepts and Definitions

The integral of a complex function along a curve or surface is called a complex integral. This definition arises because, unlike real functions that move along the real axis in various positive and negative directions, complex functions move in the complex plane. Therefore, a complex integral is analogous to a real function's line integral, with the definition based on the path of the curve.

To proceed, we use a partitioning approach. Let the curve C in the plane have start point A and endpoint B. If the function $f(z) = u(x,y) + iv(x,y)$ is defined on C, we partition the curve into n segments, where each segment corresponds to a point $z_k = x_k + iy_k$, and $\Delta z_k = \Delta x_k + i\Delta y_k = z_{k+1} - z_k$. For each segment $z\widehat{z_{k+1}}$, we choose a point $\zeta_k = \xi_k + i\eta_k$, resulting in the sum:

$$S(f, z_0, z_1, ..., z_n) = \sum_{k=0}^{n} f(\zeta_k)\Delta z_k$$

Next, we take the limit as the width of each partition tends to zero. Let $\sigma = \max_{0\le k\le n}(\Delta z_k)$, and as $\sigma \to 0$, the complex integral of $f(z)$ along C from A to B is defined as:

$$\int_C f(z)\mathrm{d}z = \lim_{\sigma\to 0}\sum_{k=0}^{n} f(\zeta_k)\Delta z_k$$

Conversely, the complex integral of $f(z)$ along C from B to A is:

$$\int_{C^-} f(z)\mathrm{d}z$$

A closed curve's integral is called a loop integral. If C is a closed curve, the loop integral of $f(z)$ along C is:

$$\oint_C f(z)\mathrm{d}z$$

It is important to note that a complex integral depends on the path of the curve, so we cannot simply define it in terms of intervals as we do with definite integrals.

Similar to real function line integrals, the result of a complex curve integral typically depends on the values of the function along the path, as well as the length and shape of the path. Essentially, we are integrating the surface area between the function and the curve, which, in physical terms, corresponds to the mass of the curve.

We know that if the integrand $f(z) = u(x,y) + iv(x,y)$ is continuous on the curve C, it is integrable. Under this assumption, by separating the real and imaginary parts of the complex function, we can derive the following:

$$
\begin{aligned}
S(f, z_0, z_1, ..., z_n) &= \sum_{k=0}^{n} f(\zeta_k)\Delta z_k \\
&= \sum_{k=0}^{n}(u_k + iv_k) \cdot (\Delta x_k + i\Delta y_k) \\
&= \sum_{k=0}^{n}(u_k \Delta x_k + u_k i \Delta y_k + iv_k \Delta x_k - v_k \Delta y_k) \\
&= \sum_{k=0}^{n}(u_k \Delta x_k - v_k \Delta y_k) + i\sum_{k=0}^{n}(v_k \Delta x_k + u_k \Delta y_k)
\end{aligned}
$$

As $\sigma \to 0$, we have $\max_{0 \leq k \leq n} |\Delta x_k| \to 0$ and $\max_{0 \leq k \leq n} |\Delta y_k| \to 0$, so the complex integral becomes:

$$\int_C f(z)\mathrm{d}z = \int_C u\mathrm{d}x - v\mathrm{d}y + i\int_C v\mathrm{d}x + u\mathrm{d}y$$

This is familiar, as it is simply the sum of two real-valued line integrals of the second kind.

4.1.1 Basic Properties

Similar to line integrals, here are some obvious properties of complex integrals.

$$\int_C kf(z)\mathrm{d}z = k\int_C f(z)\mathrm{d}z$$

$$\int_C [f(z) \pm g(z)]\mathrm{d}z = \int_C f(z)\mathrm{d}z \pm \int_C g(z)\mathrm{d}z$$

$$\int\limits_{C_1} f(z)\mathrm{d}z + \int\limits_{C_2} f(z)\mathrm{d}z = \int\limits_{C_1+C_2} f(z)\mathrm{d}z$$

$$\left|\int_C f(z)\mathrm{d}z\right| \leq \int_C f(z)\mathrm{d}z$$

Additionally, the direction of the curve affects the sign of the integral:

$$\int_{C^-} f(z)\mathrm{d}z = -\int_C f(z)\mathrm{d}z$$

Finally, the inequality for controlling the complex integral. Let C be a curve of length L and $M > 0$ be a constant. If $|f(z)| \leq M$, then:

$$\left|\int\limits_C f(z)dz\right| \leq \int\limits_C |f(z)|ds \leq ML$$

4.2 Topological Concepts Supplement

Let's first get acquainted with some basic topological concepts to help explain the definitions that follow.

The ***winding number*** of a closed curve in the plane around a point indicates how many times the curve winds around that point. This number is direction-dependent, meaning that if the curve winds counterclockwise, the winding number is positive, and if it winds clockwise, the winding number is negative. This can also be calculated using the change in the argument (angle):

$$w = \frac{1}{2\pi}\Delta\theta \arg z$$

Consider topological spaces X, Y and two continuous functions $f, g : X \to Y$. There exists a continuous mapping $H(x,t) : X \times [0,1] \to Y$ where $x \in X$, $t \in [0,1]$, and for all x, we have:

$$f(x) = H(x,0)$$
$$g(x) = H(x,1)$$

In this case, we say that f and g are **homotopic**, denoted as $f \simeq g$. Conceptually, this means that the continuous mappings from two spaces can deform continuously into one another, and thus two paths are homotopic if and only if their winding numbers are the same. This is only a rough understanding for now.

If there are no non-empty disjoint open sets in a topological space, meaning any two points in the space can be connected by a continuous path and any two closed paths (referred to as closed curves later) in the space are homotopic, then we call the space **simply connected**. Intuitively, in a simply connected space, all closed curves can continuously shrink to a point.

A **Jordan curve** (also known as a simple closed curve) refers to a non-self-intersecting closed curve in the plane. ***Compared to the commonly used "x homologous to 0 inside y" in Lars Ahlfors' textbooks, which generally emphasizes that "the enclosed region has no holes," I prefer to use the Jordan curve definition. While I really want to write about some topological homology theory, I will hold off for now.***

A quick side note: in my personal opinion, having at least a basic conceptual understanding of fundamental concepts in topology (especially algebraic topology) and abstract algebra is a necessary foundation before studying real analysis. This way, you can not only better relate concepts and definitions but also handle those foreign textbooks that love to use various topology-related terms to describe concepts. *It's known that many foreign courses start with topology, and math department textbooks tend to be more abstract compared to textbooks in non-math departments in China.*

4.3 Cauchy Integral Theorem

4.3.1 Definition and Riemann's Proof

Here is the common textbook definition: for an analytic function $f(z)$ defined on a simply connected region D, the line integral of $f(z)$ along any closed, piecewise smooth curve in D is zero.

Riemann's proof is very simple and is often used in textbooks as a typical example of proof. Since $f(z)$ is analytic on D, $f'(z)$ exists. The additional condition required for this proof is that $f'(z)$ is continuous within D.

Let's recall Green's theorem. For two functions $P(x, y)$ and $Q(x, y)$ with continuous first partial derivatives in a closed region D, consider the boundary curve C of D. We have:

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}x\mathrm{d}y = \oint_C (P\mathrm{d}x + Q\mathrm{d}y)$$

Applying the previous transformations and Green's theorem:

$$\begin{aligned} \int_C f(z)\mathrm{d}z &= \int_C u\mathrm{d}x - v\mathrm{d}y + i\int_C v\mathrm{d}x + u\mathrm{d}y \\ &= \iint_D \left(-\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) \mathrm{d}x\mathrm{d}y + i\iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) \mathrm{d}x\mathrm{d}y \end{aligned}$$

Since $f(z)$ is analytic, the Cauchy-Riemann equations (CRE) hold. We know that $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, so:

$$\iint_D \left(-\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) \mathrm{d}x\mathrm{d}y = 0, \quad \iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) \mathrm{d}x\mathrm{d}y = 0$$

Thus, we have:

$$\int_C f(z)\mathrm{d}z = 0$$

4.3.2 Path Independence Principle

To extend the Cauchy Integral Theorem to multiply connected regions, we use the following more general definition:

Consider a region $D \in \mathbb{C}$ and a function $f : D \to \mathbb{C}$, with two points $x, y \in D$ connected by two piecewise smooth curves C_1, C_2. If C_1 and C_2 are homotopic (i.e., they can be continuously deformed into one another within D), then:

$$\int_{C_1} f(z)\mathrm{d}z = \int_{C_2} f(z)\mathrm{d}z$$

Thus, we do not need to restrict the choice of path, but only need to consider the homotopy class of the paths.

Specifically, if C is a piecewise smooth closed curve that is homotopic to a constant map (i.e., it can be continuously deformed into a single point), then:

$$\oint_C f(z)\mathrm{d}z = 0$$

Some textbooks refer to the fact that the integral of an analytic function along a closed curve in a region does not change under continuous deformations of the curve within the region as the "Path Independence Principle."

4.3.3 The Composite Closed Path Theorem

The Composite Closed Path Theorem is an important corollary of the Cauchy Integral Theorem. Consider a multiply connected domain D and a closed curve C in D that encloses multiple closed curves $C_1, C_2, \ldots, C_n$ within C, where the C_k's do not intersect each other. Consider an analytic function $f(z)$ in D, and define $\Gamma = C_1 + C_2 + \ldots + C_n$. Then:

$$\oint_{\Gamma} f(z)\mathrm{d}z = 0$$

and also:

$$\oint_{C} f(z)\mathrm{d}z = \sum_{k=1}^{n} \oint_{C_k} f(z)\mathrm{d}z$$

4.4 Cauchy Integral Formula

Consider a simply connected open set $U \in \mathbb{C}$ and an analytic function $f : U \to \mathbb{C}$. For a positively oriented Jordan curve C in U and a point z_0 inside C, we have:

$$f(z_0) = \frac{1}{2\pi i} \oint_{C^+} \frac{f(z)}{z - z_0}\,\mathrm{d}z$$

Proof. Consider the function $F(z) = \frac{f(z)}{z-z_0}$, which is analytic everywhere except at z_0. Let R be a circle of radius $r > 0$ centered at z_0, such that R and its interior lie within U. Applying the generalized Cauchy Integral Theorem for multiply connected regions to the region enclosed by R and C, we get:

$$\oint_{C} F(z)\,\mathrm{d}z = \oint_{R} F(z)\,\mathrm{d}z$$

Since $f(z)$ is analytic, the above equation holds for any small r. We now focus on shrinking the radius r to zero, as the curve R approaches z_0. We need to prove that:

$$\lim_{r\to 0} \int_{R} F(z)\,\mathrm{d}z = 2\pi i f(z_0)$$

Parameterize the path by $z - z_0 = re^{i\theta}$, where $0 \leq \theta \leq 2\pi$. Then, $\mathrm{d}z = ire^{i\theta}\,\mathrm{d}\theta$, so we get:

$$\begin{aligned}
\lim_{r\to 0}\oint_R F(z)\,\mathrm{d}z &= \lim_{r\to 0}\int_0^{2\pi}\frac{f(z_0+re^{i\theta})}{re^{i\theta}}\cdot re^{i\theta}i\,\mathrm{d}\theta \\
&= \int_0^{2\pi}\lim_{r\to 0} f(z_0+re^{i\theta})i\,\mathrm{d}\theta \\
&= \int_0^{2\pi} f(z_0)i\,\mathrm{d}\theta \\
&= 2\pi i f(z_0)
\end{aligned}$$

Substituting this result into the Cauchy Integral Formula, the equality holds. ■

4.4.1 Maximum Modulus Principle

The Maximum Modulus Principle states that if a function is analytic and non-constant within a bounded region, then the modulus of the function does not achieve a local maximum inside the region. In other words, the maximum modulus must occur on the boundary.

4.4.2 Higher-Order Derivative Formula

An analytic function is infinitely differentiable. By extending the Cauchy Integral Formula to higher derivatives, we obtain the higher-order derivative formula, which allows us to compute derivatives using integrals, or vice versa:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\,\mathrm{d}z$$

4.5 Cauchy Inequality and Liouville's Theorem

By applying the higher-order derivative formula, we can derive further conclusions.

If a function is analytic in a region and its modulus has an upper bound on the boundary of the region, then the modulus of the function is also bounded within the region and cannot exceed the bound on the boundary.

Consider an analytic function $f(z)$ on a region D with $|f(z)| \leq M$. For any $z_0 \in D$, draw a circle centered at z_0 with radius L such that the circle and its interior lie within D. Then, we have:

$$|f^{(n)}(z_0)| \leq \frac{n!M}{L^n}$$

The proof is straightforward by applying the absolute value inequality:

$$|f^{(n)}(z_0)| = \left|\frac{n!}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\,\mathrm{d}z\right| |\mathrm{d}z| \leq \frac{n!}{2\pi i}\cdot\frac{M}{L^{n+1}}\cdot 2\pi L$$

Thus:

$$|f^{(n)}(z_0)| \leq \frac{n!M}{L^n}$$

This proves the Cauchy Inequality.

Another famous result is Liouville's Theorem, which states that if an analytic function $f(z)$ is bounded on the entire complex plane, then it must be constant. The proof follows from the Cauchy Inequality applied to the first derivative of the function. Consider any $z_0 \in \mathbb{C}$ and any $L > 0$, where $f(z)$ is analytic within the disk $|z - z_0| < L$, and suppose $|f(z)| \leq M$. By the Cauchy Inequality, we get:

$$|f'(z_0)| \leq \frac{M}{L}$$

If an analytic function is not constant on the entire complex plane, then by Liouville's Theorem, it must be unbounded.

5 Power Series Expansion of Analytic Functions

An analytic function can be expanded into an infinite series, and this series converges to the function within its domain. This series is known as a convergent power series, and its properties are crucial in the study of analytic functions.

5.1 Taylor Series

As we know, a power series within a so-called radius of convergence represents an analytic function. Therefore, we are interested in how to express a function as a complex power series. Functions that can be represented as power series are called *analytic functions*.

In real analysis, we study the Taylor series of an infinitely differentiable real function to approximate it near a point. Similarly, in complex analysis, for an analytic function near a point z_0, its Taylor series is given by:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

The condition for a function to be analytic at a point is that it can be expanded into a power series in a neighborhood around that point. This is why power series play such an important role in the study of analytic functions. However, this condition is quite stringent in practice.

The remainder of the n-th order Taylor series of the function $f(z)$ is given by:

$$R(z) = f(z) - \sum_{k=0}^{n} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k$$

For $f(z)$ to be expandable into a power series in a neighborhood of z_0, it is necessary and sufficient that the remainder of its Taylor series converges to 0 in that neighborhood.

5.1.1 Taylor's Theorem

Based on the properties of complex power functions, consider an analytic function $f(z)$ in a domain D. For $z_0 \in D$, there exists an open disk $O = \{z : |z - z_0| < R, R > 0\} \subset D$ where the function can be uniquely expanded as:

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

By generalizing the Cauchy Integral Formula, we can express the Taylor coefficients c_n as:

$$c_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta$$

where C is a positively oriented, closed curve centered at z_0 with radius less than R, lying entirely within the region D. The integral represents the contour integral around C. The Taylor expansion of a function $f(z)$ in D is a necessary and sufficient condition for $f(z)$ to be analytic in D.

5.2 Laurent Series

The Laurent series is a power series that is expanded in a ring-shaped region in the complex plane, including both positive and negative powers of $(z - z_0)$. The region of convergence for a Laurent series is typically an annular region, referred to as the *annular region of convergence*. A Laurent series that converges at both boundaries of its region of convergence is known as a *bilateral power series*.

Laurent series are particularly useful in representing analytic functions on an annulus. Consider an analytic function $f(z)$ defined in the annular region $Q : R_1 < |z - z_0| < R_2$, where the function has a unique Laurent expansion:

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$$

For $R_1 < R_2$, the series converges absolutely and uniformly within the annulus Q, and $f(z)$ is analytic in Q. The region Q is called the *annular region of convergence* of the Laurent series. If $R_1 > R_2$, the series diverges everywhere.

Typically, we focus on the negative power terms of the Laurent series, known as the *principal part* of $f(z)$:

$$f(z) = \sum_{n=1}^{\infty} c_{-n} (z - z_0)^{-n}$$

The remaining non-negative power terms are referred to as the *regular part* of $f(z)$:

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

For a Laurent series, the center of the annular region of convergence is often chosen to be the singularity of the function, which is crucial in residue theory.

5.2.1 Laurent's Theorem

For an analytic function $f(z)$ defined on an annular region $Q : R_1 < |z-z_0| < R_2$, the function can be expanded into a Laurent series. The coefficients c_n of the Laurent series are defined as:

$$c_n = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, \mathrm{d}\zeta$$

Thus, in the annular region $Q : R_1 < |z-z_0| < R_2$, an analytic function $f(z)$ can be expressed as a two-sided power series:

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$$

Proof. Let z be a point inside the annular region $Q : R_1 < |z - z_0| < R_2$. To prove Laurent's theorem, we decompose the function $f(z)$ into two functions $g(z)+h(z)$, and show that both $g(z)$ and $h(z)$ have removable singularities at ∞, where $g(z)$ can be expanded as a series of non-negative powers of $(z - z_0)$ and $h(z)$ can be expanded as a series of non-negative powers of $\frac{1}{(z-z_0)}$.

We define two circles C_1 and C_2, with $R_1 < r_1 < |z - z_0| < r_2 < R_2$:

$$C_1 : |\zeta - z_0| = r_1 \quad \text{and} \quad C_2 : |\zeta - z_0| = r_2$$

The exact values of r_1 and r_2 do not matter, as long as they satisfy $R_1 < r_1 < |z-z_0| < r_2 < R_2$, because the integrals over these paths do not depend on the specific choice of r_1 and r_2. We apply the Cauchy Integral Formula to obtain:

$$g(z) = \frac{1}{2\pi i} \oint_{C_2} \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta$$

$$h(z) = -\frac{1}{2\pi i} \oint_{C_1} \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta$$

For $g(z)$ and $|z - z_0| < r_2 < R_2$, we apply the Taylor series expansion directly:

$$c_n = \frac{1}{2\pi i} \oint_{C_2} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, \mathrm{d}\zeta$$

Thus, we can expand $g(z)$ as:

$$g(z) = \frac{1}{2\pi i} \oint_{C_2} \frac{f(\zeta)}{\zeta - z_0} \, \mathrm{d}\zeta = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

For $h(z)$ and $R_1 < r_1 < |z - z_0|$, we rewrite the integrand:

$$\frac{f(\zeta)}{\zeta - z} = \frac{f(\zeta)}{(\zeta - z_0) - (z - z_0)} = \frac{f(\zeta)}{(\zeta - z_0)} \left[\frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} \right]$$

Next, we expand the geometric series for the second factor:

$$\frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} = \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0} \right)^n$$

Substituting this into the expression for $h(z)$, we get:

$$\begin{aligned} h(z) &= -\frac{1}{2\pi i} \oint_{C_1} \frac{f(\zeta)}{(\zeta - z_0)} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0} \right)^n \mathrm{d}\zeta \\ &= \sum_{n=0}^{\infty} (z - z_0)^{-n} \left(\frac{1}{2\pi i} \oint_{C_1} \frac{f(\zeta)}{(\zeta - z_0)^{-n+1}} \mathrm{d}\zeta \right) \end{aligned}$$

By applying the Cauchy Integral Formula to the inner integral, we find:

$$c_{-n} = \frac{1}{2\pi i} \oint_{C_1} \frac{f(\zeta)}{(\zeta - z_0)^{-n+1}} \mathrm{d}\zeta$$

Thus, $h(z)$ can be expanded as:

$$h(z) = \sum_{n=0}^{\infty} c_{-n} (z - z_0)^{-n}$$

Combining $g(z)$ and $h(z)$, we obtain the Laurent series for $f(z)$:

$$f(z) = g(z) + h(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n + \sum_{n=0}^{\infty} c_{-n} (z - z_0)^{-n}$$

Thus, we have the two-sided Laurent series expansion:

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$$

Proof complete. ■

The meaning is obvious. The Laurent series is a generalization of the Taylor series. The Taylor series is only applicable to the local expansion of an analytic function at a specific point, which allows us to learn only about the behavior of the function near that point. On the other hand, the Laurent series overcomes this limitation by allowing us to expand the function within an annular region, making it suitable for regions that include singularities.

By the way, interested readers can refer to *Lars Ahlfors*'s book for the proof of the Laurent theorem. The first part of the reasoning is similar, but the transformation of $h(z)$ is quite elegant (the method used in this article is the substitution method that most textbooks will use).

6 Residue Theory

The residue, or "the number that remains," refers to the coefficient of the negative powers in the Laurent series expansion of a complex function near a singularity. It is an important characteristic of a complex function at its isolated singularity. Residue theory primarily involves using Laurent series to study the properties around isolated singularities, and it is further applied in techniques for evaluating integrals. It is an essential skill for advancing in integral calculus.

6.1 Isolated Singularity

A function $f(z)$ is said to have an isolated singularity at z_0 if it is not analytic at z_0, but is analytic in some punctured neighborhood $0 < |z - z_0| < \delta$.

If a function $f(z)$ has an isolated singularity at z_0, then $f(z)$ can be expanded in a Laurent series in some punctured neighborhood of z_0:

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$$

In practice, we focus on the negative powers of the Laurent series $f(z) = \sum_{n=1}^{\infty} c_{-n}(z - z_0)^{-n}$, which is called the principal part of $f(z)$, while the remaining non-negative part $f(z) = \sum_{n=0}^{\infty} c_n(z - z_0)^n$ is called the regular part of $f(z)$.

Removable Singularity

If we can assign a value $f(z_0) = C_0$ such that $f(z)$ becomes analytic in the entire neighborhood δ, then the singularity at z_0 is called a removable singularity. The typical way to resolve this situation is by defining the function at the singularity.

The Laurent series of $f(z)$ in the neighborhood of a removable singularity does not have a principal part. If the isolated singularity at z_0 is removable, then $\lim_{z\to z_0} f(z) = c_0$.

A removable singularity is not truly considered a singularity since the limit exists and its residue is zero. Therefore, in general, there is no need to calculate the residue at a removable singularity.

Pole

If $f(z)$ has a Laurent series expansion at z_0 in some punctured neighborhood δ, and there are finitely many terms with $n < 0$ such that $c_n \neq 0$, then z_0 is a pole of $f(z)$.

At a pole z_0, the Laurent series of $f(z)$ in the neighborhood δ has a finite number of terms in the principal part. If the isolated singularity at z_0 is a pole, then $\lim_{z\to z_0} f(z) = \infty$.

Essential Singularity

At an essential singularity, the Laurent series of the function has infinitely many terms with negative powers (how dreadful, right?). Specifically, if $f(z)$ has infinitely many terms with $n < 0$ such that $c_n \neq 0$ in its Laurent series expansion around z_0, then z_0 is an essential singularity of $f(z)$.

If z_0 is an essential singularity of $f(z)$, then $\lim_{z\to z_0} f(z)$ does not exist, which is both a necessary and sufficient condition.

6.2 Residue

Let $f(z)$ be analytic inside and on a Jordan curve C. By Cauchy's Integral Theorem, we know that $\oint_C f(z)\,dz = 0$, which is a simple result. However, if $f(z)$ has isolated singularities inside C, we cannot apply this theorem directly and must seek another method to evaluate the integral.

Consider a unique value R such that $f(z) - \frac{R}{z-z_0}$ is analytic in some punctured neighborhood $0 < |z - z_0| < \delta$. This defines the residue of $f(z)$ at z_0, and this condition ensures that there are no singularities in the neighborhood of z_0. The residue is denoted $\text{Res}(f(z), z_0)$, and in cases of no ambiguity, it can also be written as $\text{Res}(z_0)$.

Let z_0 be an isolated singularity of $f(z)$. The Laurent series expansion of $f(z)$ at z_0 is:

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$$

To calculate the residue, we generally take the coefficient of the negative power term c_{-1}, as it is simpler. From the definition of the coefficients in the Laurent series (essentially an extension of Cauchy's Integral Formula):

$$c_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}}\,dz$$

Substituting gives:

$$\text{Res}(z_0) = c_{-1} = \frac{1}{2\pi i} \oint_C f(z)\,dz$$

Thus,

$$\oint_C f(z)\,dz = \text{Res}(z_0) 2\pi i$$

Note that, by definition, we should not attempt to compute the residue of non-isolated singularities. For example, if there are infinitely many singularities that are not isolated, we cannot perform a Laurent expansion and thus cannot compute the residue.

6.2.1 Residue Theorem

Let us first write a more general definition. Consider a simply connected open set D, and suppose that the function $f(z)$ is analytic in D except for a finite number of isolated singularities. A closed curve C inside D encloses these singularities, and C does not pass through any singularities. Let $w(z_k)$ denote the winding number of C around z_k, then we have:

$$\oint_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} w_k \,\text{Res}(z_k)$$

However, if C is a Jordan curve, this implies that all w_k are equal to 1, so we get:

$$\oint_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(z_k)$$

6.2.2 Residues at Finite Points

- The residue of a function at a removable singularity is 0.
- The residue at an essential singularity requires expanding the function into a Laurent series and calculating the coefficient c_{-1}, as defined earlier.

For poles, besides using the Laurent series expansion, there is also a more commonly used special method:

I Here is another necessary and sufficient condition to determine whether an isolated singularity is a pole. Consider a function $\lambda(z) \neq 0$ that is analytic in a neighborhood δ of z_0, then z_0 is a pole of order m of $f(z)$:

$$f(z) = \lambda(z)\frac{1}{(z-z_0)^m}$$

According to the definition, we have:

$$\mathrm{Res}(f(z), z_0) = \frac{\lambda^{m-1}(z_0)}{(m-1)!}$$

Clearly, if $m = 1$, this simplifies to:

$$\mathrm{Res}(f(z), z_0) = \lambda(z_0)$$

II Another common form is $f(z) = \frac{P(z)}{Q(z)}$, where $P(z)$ and $Q(z)$ are analytic in a neighborhood δ of z_0, and z_0 is a simple zero of $Q(z)$ with $P(z) \neq 0$. In this case, we have (using L'Hopital's rule):

$$\mathrm{Res}(f(z), z_0) = \lim_{z \to z_0} (z - z_0)\frac{P(z)}{Q(z)} = \frac{P(z_0)}{Q'(z_0)}$$

6.3 Logarithmic Residue

Suppose a function $f(z)$ is analytic and non-zero on a closed curve C, and meromorphic inside C (note: this term has not been used in the text before, it means the function inside the region has no singularities except for removable singularities and poles, which is a weaker property than analyticity). If the integral is of the form:

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz$$

Since $\frac{f'(z)}{f(z)}\,dz = d\ln f(z)$, we call this integral the logarithmic residue of $f(z)$ on C. Let the number of zeros of $f(z)$ inside C be Z, and the number of poles be P, where zeros and poles are counted with multiplicity. Then we have:

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = Z - P$$

This is cool! The logarithmic residue reflects the relationship between the number of zeros and poles inside the closed curve, and this concept will be extended later.

6.3.1 Argument Principle

Using the same definition, suppose that $f(z)$ is analytic and non-zero on a closed curve C, and meromorphic inside C. Let the change in the argument of $f(z)$ along C be denoted by $\Delta_C \arg f(z)$, then we have:

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = \frac{\Delta_C \arg f(z)}{2\pi} = Z - P$$

Specially, if $f(z)$ has no singularities inside C (i.e., $f(z)$ is analytic inside C), the above equation simplifies to:

$$\frac{\Delta_C \arg f(z)}{2\pi} = N$$

6.3.2 Rouche's Theorem

Let $f(z)$ and $g(z)$ be analytic on and inside a Jordan curve C, and suppose that $|f(z)-g(z)| < |f(z)|$ on C, then the number of zeros of $f(z)$ and $f(z)+g(z)$ inside C are the same.

There is an interesting discussion on Rouche's Theorem where the analogy is made to a dog on a leash walking around a tree (with the tree treated as a point and the person not touching the tree). As long as the length of the leash is always shorter than the distance from the person to the tree, the dog will never make an extra loop around the tree. Thus, the number of loops the dog makes equals the number of loops the person makes around the tree.

We can extend this further. If the function is meromorphic inside C and the other conditions are the same, then:

$$Z(f(z)) - P(f(z)) = Z(f(z)+g(z)) - P(f(z)+g(z))$$

References

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