Introduction to Laplace's Equation

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1 Introduction

The Laplace equation is a type of partial differential equation, also known as the potential equation or harmonic equation. It holds a pivotal position in mathematics and physics, appearing frequently in familiar areas like the theory of harmonic functions in complex analysis, Fourier analysis, and variational methods.

Recently, my work has touched upon related concepts, so I decided to write a brief article on the topic. This article will provide a concise overview of the definition and basic form of the Laplace equation, along with its common, straightforward solutions and applications.

2 Basic Concepts

Let's start by discussing the Laplace operator.

We know that the gradient is a vector field that represents the maximum rate of change of a function in various directions. In an n-dimensional Euclidean space, the gradient of a smooth scalar function is defined as

$$\nabla f = \left\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right\rangle$$

The Laplace operator, denoted as Δ, is a second-order differential operator in n-dimensional Euclidean space. For a smooth scalar function f, the Laplace operator is defined as the **divergence of the gradient**

$$\Delta f = \nabla \cdot \nabla f = \nabla^2 f$$

Below are the forms of the Laplace operator in two-dimensional and three-dimensional spaces under common coordinate systems. *These are very fundamental and commonly used, and readers should memorize them.*

2.0.1 Two-Dimensional Space

In two-dimensional space, the representation of the Laplace operator in Cartesian coordinates is given by

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

In polar coordinates, it is represented as

$$\Delta f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}$$

In parabolic coordinates, the expression is

$$\frac{1}{u\nu\left(u^2+\nu^2\right)}\left[\frac{\partial}{\partial u}\left(u\nu\frac{\partial f}{\partial u}\right) + \frac{\partial}{\partial \nu}\left(u\nu\frac{\partial f}{\partial \nu}\right)\right] + \frac{1}{u^2\nu^2}\frac{\partial^2 f}{\partial \theta^2}$$

2.0.2 Three-Dimensional Space

In three-dimensional space, the representation of the Laplace operator in Cartesian coordinates is

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

In cylindrical coordinates, it is expressed as

$$\Delta f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$$

In parabolic cylindrical coordinates, it is given by

$$\frac{1}{u^2+v^2}\left(\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2}\right) + \frac{\partial^2 f}{\partial z^2}$$

In spherical coordinates, it is represented as

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin^2\phi}\frac{\partial^2 f}{\partial \theta^2} + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi\frac{\partial f}{\partial \phi}\right)$$

2.0.3 Symmetry and Self-Adjointness

To review, a linear operator A is symmetric if, in a Hilbert space, it satisfies the condition that for all $f, g \in \mathrm{Dom}(A)$, we have $\langle Af, g\rangle = \langle f, Ag\rangle$. A symmetric operator is a weaker condition than being self-adjoint; it only requires maintaining inner product symmetry within its domain and does not necessitate being self-adjoint over the entire space. For the Laplace operator Δ, we typically define it on an appropriate function space, such as $H_0^1(\Omega)$ within the L^2 space, where Ω is a bounded region. In this case, the Laplace operator Δ is symmetric.

A linear operator A is self-adjoint if and only if it is symmetric and its domain coincides with the domain of its adjoint operator. Under typical boundary conditions, such as Dirichlet boundary conditions ($f = 0$ on the boundary), the Laplace operator is self-adjoint. This is because such boundary conditions ensure that the domain of Δ and the domain of its adjoint operator are the same.

Some operators may not be self-adjoint on their initial domain but can be made self-adjoint by extending their domain. A common simple example in Hilbert space involves the classic method of self-adjoint extension that utilizes deficiency spaces and deficiency indices. For a symmetric operator, we define the deficiency space as

$$\mathcal{K}_{\pm} = \ker\left(A^* \mp iI\right)$$

The dimension of the deficiency space, denoted as $d_{\pm} = \dim(\mathcal{K}_{\pm})$, is referred to as the deficiency index of the operator. There are three cases to consider

1. If $d_+ = d_- = 0$, then A is self-adjoint and does not require extension. 2. If $d_+ = d_-$, then A may potentially become self-adjoint through some extension. The specific extension will depend on different boundary conditions. 3. If $d_+ \neq d_-$, then A cannot be extended to a self-adjoint operator.

As for more specific analyses and examples of extensions, there may be opportunities to discuss them in detail later in a functional analysis column on linear operator theory. Now, let's continue to examine the Laplace equation.

2.1 Definition

The Laplace equation pertains to solving real functions f that are twice continuously differentiable up to the boundary with respect to real variables x, y, z, in the form of

$$\Delta f = 0$$

Solutions to the Laplace equation are referred to as harmonic functions. Harmonic functions possess excellent properties, which are familiar from calculus, and we will discuss their properties and related theorems in detail in a later article on complex analysis.

2.1.1 Potential Field

On a related note, a potential field is a scalar or vector field where the value at each point in space represents a certain "potential," such as gravitational potential, electric potential, or temperature field. A potential field is typically defined by a scalar function; for example, in physics, we have a potential function ψ, and the field strength can be expressed as

$$\vec{F} = -\nabla\psi$$

This aspect is not crucial. What is important is that when describing fields in regions with no sources (or distributions of charge, mass, etc.), the potential function usually satisfies the Laplace equation. This implies that in a source-free electrostatic field or gravitational field, the potential distribution in space remains unchanged, exhibiting a stable state. Therefore, potential functions that satisfy the Laplace equation are referred to as harmonic functions and have no local extrema within their domain.

2.2 Boundary Value Problems and Exterior Problems

The distinction between boundary value problems (sometimes referred to as interior problems) and exterior problems in the context of the Laplace equation primarily lies in the definition of the region of interest and the different boundary conditions imposed.

We commonly use boundary value problems when studying physical phenomena within closed or semi-closed systems, such as fluid flow inside pipes, temperature distribution within buildings, or electric fields in insulated regions. Boundary value problems typically involve a finite region where boundary conditions must be defined on the boundary of that finite area.

In contrast, exterior problems deal with infinite or semi-infinite regions, usually the entire space minus a bounded region. For instance, this includes studying fields outside a region in infinite space, such as the electric field distribution of a charge in an infinite space, the propagation of sound or heat sources outside an open area, or situations in acoustics, heat conduction, or static electric fields where solutions often require that the fields decay to zero or approach a constant at infinity. The boundary conditions for exterior problems are usually specified at infinity, with common cases being Dirichlet exterior problems and Neumann exterior problems.

To differentiate between the two types of problems, we sometimes refer to the first and second boundary value problems as Dirichlet interior problems and Neumann interior problems, respectively.

2.3 General Solution

Let's discuss the general solution to the Laplace equation.

The general solution to the homogeneous Laplace equation can be obtained using the method of separation of variables. We will consider the case in three-dimensional space and discuss the situations in different coordinate systems.

2.3.1 Cartesian Coordinates

Given the Laplace equation

$$\Delta f(x,y,z) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

The solution can be expressed in a separable form

$$f(x,y,z) = X(x)Y(y)Z(z)$$

Substituting this into the equation gives

$$\begin{aligned}\frac{\partial^2 X(x)}{\partial x^2}Y(y)Z(z) + \frac{\partial^2 Y(y)}{\partial y^2}X(x)Z(z) + \frac{\partial^2 Z(z)}{\partial z^2}X(x)Y(y) = 0 \\ \frac{\partial^2 X(x)}{\partial x^2}\frac{1}{X(x)} + \frac{\partial^2 Y(y)}{\partial y^2}\frac{1}{Y(y)} + \frac{\partial^2 Z(z)}{\partial z^2}\frac{1}{Z(z)} = 0\end{aligned}$$

Since x, y, z are independent variables, each term must equal a constant. Introducing separation constants $-k_x^2, -k_y^2, -k_z^2$, we find the general solution:

$$\begin{aligned}
X(x) &= A_1 \cos(k_x x) + A_2 \sin(k_x x) \\
Y(y) &= B_1 \cos(k_y y) + B_2 \sin(k_y y) \\
Z(z) &= C_1 \cos(k_z z) + C_2 \sin(k_z z)
\end{aligned}$$

Alternatively, this can be expressed simply as

$$f(x, y, z) = e^{\pm i k_x x} e^{\pm i k_y y} e^{\pm i k_z z}$$

2.3.2 Cylindrical Coordinates

The Laplace operator in cylindrical coordinates is given by

$$\begin{aligned}
\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2} &= 0 \\
\frac{1}{rR}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + \frac{1}{r^2 f}\frac{\partial^2 f}{\partial \theta^2} + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} &= 0
\end{aligned}$$

Using the method of separation of variables, we introduce separation constants $-k_r^2, -k_\theta^2, -k_z^2$. Breaking the equation into independent components:

$$\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = k_z^2, \quad \frac{1}{rR}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) = k_r^2, \quad \frac{1}{r^2 f}\frac{\partial^2 f}{\partial \theta^2} = k_\theta^2$$

For $Z(z)$:

$$\frac{\partial^2 Z}{\partial z^2} - k_z^2 Z = 0, \quad Z(z) = Ae^{k_z z} + Be^{-k_z z}$$

For $R(r)$:

$$\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) - k_r^2 r R = 0, \quad R(r) = CJ_m(k_r r) + DY_m(k_r r)$$

where J_m and Y_m are Bessel functions of the first and second kinds.

For $f(\theta)$:

$$\frac{\partial^2 f}{\partial \theta^2} + k_\theta^2 f = 0, \quad f(\theta) = E\cos(k_\theta \theta) + F\sin(k_\theta \theta)$$

The general solution to the Laplace equation in cylindrical coordinates is

$$\begin{aligned}
f(r, \theta, z) &= R(r) f(\theta) Z(z) \\
&= (CJ_m(k_r r) + DY_m(k_r r))\,(E\cos(k_\theta \theta) + F\sin(k_\theta \theta))\,\left(Ae^{k_z z} + Be^{-k_z z}\right)
\end{aligned}$$

2.4 Analytic Functions

In complex analysis, we have learned the definition of analytic functions, which refers to functions that are differentiable at every point in a given region. If a complex function $f(z) = u(x,y) + iv(x,y)$ satisfies the Cauchy-Riemann equations in a certain region, then it is analytic

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

As previously mentioned, if a real function $u(x,y)$ satisfies the Laplace equation (i.e., $\Delta u = 0$) in a region, then u is a harmonic function. If we assume that $u(x,y)$ is a harmonic function, there exists a unique function $v(x,y)$ such that $f(z) = u(x,y) + iv(x,y)$ is analytic. This function v also satisfies the Laplace equation.

Thus, we can conclude that if a function is analytic, then both its real and imaginary parts satisfy the Laplace equation.

3 Classical Solutions

3.1 Separation of Variables (Boundary Conditions)

Just like in the previous derivation, in the method of separation of variables, we assume that the solution can be expressed as a product of functions, each depending on a single independent variable. This method transforms the original partial differential equation into several ordinary differential equations, simplifying the problem-solving process. This method is fundamental, and I have already discussed the separation of variables for ordinary differential equations in last year's article on differential equations, so I will skip that here.

However, when solving specific problems, we often encounter boundary conditions that further determine the values of constants k and the coefficients in the solution.

3.1.1 Dirichlet Boundary Conditions

The Dirichlet boundary condition, also known as the "first type boundary condition," specifies known values of the solution at the boundary and is commonly used to describe situations where the boundary temperature, electric potential, or displacement is fixed.

Formally, for the unknown function u, the boundary $\partial\Omega$ of the defined region Ω, and the fixed value f imposed on the boundary, the Dirichlet boundary condition is expressed as

$$u(x) = f(x), \quad x \in \partial\Omega$$

For example, in polar coordinates, consider a circular region $r \leq R$. If there is a fixed electric potential at the boundary $r = R$, the condition can be described as

$$u(R, \theta) = f(\theta)$$

To apply the method of separation of variables, we assume the solution can be expressed in the following form

$$u(r, \theta) = R(r)\Theta(\theta)$$

Substituting this form into the polar representation of Laplace's equation yields

$$\frac{1}{R(r)} \frac{d}{dr} \left(r \frac{dR}{dr} \right) + \frac{1}{\Theta(\theta)} \frac{d^2\Theta}{d\theta^2} = 0$$

Since the left side depends only on r and the right side depends only on θ, we can introduce a separation constant k

$$\frac{1}{R(r)} \frac{d}{dr} \left(r \frac{dR}{dr} \right) = -k \quad \text{and} \quad \frac{1}{\Theta(\theta)} \frac{d^2\Theta}{d\theta^2} = k$$

Thus, we obtain the equation for $R(r)$

$$r \frac{d^2R}{dr^2} + \frac{dR}{dr} + krR = 0$$

This equation is a standard Bessel equation. We can express the solution in terms of Bessel functions $J_n(x)$. Assuming $k = \left(\frac{n}{R}\right)^2$, where n is a positive integer, the solution can be written as

$$R(r) = A_n J_n \left(\frac{n}{R} r \right)$$

Next, consider the equation for $\Theta(\theta)$

$$\frac{d^2\Theta}{d\theta^2} + k\Theta = 0$$

This is a simple second-order ordinary differential equation with the solution

$$\Theta(\theta) = B_n \cos n\theta + C_n \sin n\theta$$

Combining the above results, the form of the solution is

$$u(r, \theta) = \sum_{n=0}^{\infty} \left(A_n J_n \left(\frac{n}{R} r \right) (B_n \cos n\theta + C_n \sin n\theta) \right)$$

To satisfy the boundary condition $u(R, \theta) = f(\theta)$, we substitute $r = R$

$$u(R, \theta) = \sum_{n=0}^{\infty} A_n J_n(n)(B_n \cos n\theta + C_n \sin n\theta) = f(\theta)$$

Here, $J_n(n)$ is the value of the Bessel function at n. Based on the Fourier series expansion, we can express $f(\theta)$ as a linear combination of cosine and sine terms

$$f(\theta) = \sum_{n=0}^{\infty} (D_n \cos n\theta + E_n \sin n\theta)$$

By comparing coefficients, we can establish the relationships between A_n, B_n, C_n and D_n, E_n. These coefficients can be calculated as Fourier coefficients

$$D_n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \cos n\theta \, \mathrm{d}\theta, \quad E_n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \sin n\theta \, \mathrm{d}\theta$$

Finally, the expression for the solution can be written as

$$u(r, \theta) = \sum_{n=0}^{\infty} A_n J_n \left(\frac{n}{R} r \right) (D_n \cos n\theta + E_n \sin n\theta)$$

This solution effectively describes the electric potential distribution within the circular region while satisfying the Dirichlet condition at the boundary $r = R$, $u(R, \theta) = f(\theta)$.

3.1.2 Neumann Boundary Conditions

The Neumann boundary condition, also known as the "second type boundary condition," specifies the normal derivative of the solution at the boundary. This type of condition is often used to describe situations involving flux, flow, or gradients at the boundary in physical systems.

Formally, for the unknown function u, and a specified boundary value $g(x)$ on the boundary $\partial\Omega$ of the region Ω, the Neumann boundary condition can be expressed as

$$\frac{\partial u}{\partial n} = g(x)$$

In physical terms, the normal derivative $\frac{\partial u}{\partial n}$ can represent quantities such as heat flux through a boundary.

To illustrate the use of separation of variables in solving a one-dimensional heat conduction problem, consider a one-dimensional rod of length L with temperature distribution described by the function $u(x, t)$. Suppose the initial temperature distribution is given by $u(x, 0) = f(x)$, and the rod has the following Neumann boundary conditions

- At the boundary $x = 0$, the derivative is a constant q_0 (indicating heat flux)

$$\frac{\partial u}{\partial x}(0, t) = q_0$$

- At the boundary $x = L$, the derivative is zero (indicating an adiabatic boundary)

$$\frac{\partial u}{\partial x}(L, t) = 0$$

With α representing the thermal diffusivity, the heat conduction equation is given by

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$

Assuming a separable solution of the form $u(x,t) = X(x)T(t)$ and substituting it into the heat equation yields

$$\begin{aligned} X(x)\frac{dT}{dt} &= \alpha T(t)\frac{d^2X}{dx^2} \\ \frac{1}{\alpha T}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2} &= -k \end{aligned}$$

The time-dependent equation becomes

$$\begin{aligned} \frac{dT}{dt} + \alpha kT &= 0 \\ T(t) &= Ce^{-\alpha kt} \end{aligned}$$

The spatial-dependent equation is

$$\begin{aligned} \frac{d^2X}{dx^2} + kX &= 0 \\ X(x) &= A\cos\left(\sqrt{k}x\right) + B\sin\left(\sqrt{k}x\right) \end{aligned}$$

Now, applying the boundary condition $\frac{\partial u}{\partial x}(0,t) = q_0$

$$\frac{\partial u}{\partial x}(0,t) = X'(0)T(t) = q_0$$

Calculating the derivative gives

$$X'(x) = -A\sqrt{k}\sin\left(\sqrt{k}x\right) + B\sqrt{k}\cos\left(\sqrt{k}x\right)$$

At $x = 0$, we have $X'(0) = B\sqrt{k}$, leading to

$$B\sqrt{k}T(t) = q_0 B = \frac{q_0}{\sqrt{k}T(t)}$$

Next, applying the boundary condition $\frac{\partial u}{\partial x}(L,t) = 0$

$$\frac{\partial u}{\partial x}(L,t) = X'(L)T(t) = 0$$

This implies $X'(L) = 0$

$$-A\sqrt{k}\sin\left(\sqrt{k}L\right) + B\sqrt{k}\cos\left(\sqrt{k}L\right) = 0$$

Substituting B into the equation provides a relationship between A and k

$$-A\sqrt{k}\sin\left(\sqrt{k}L\right)+\frac{q_0}{\sqrt{k}T(t)}\sqrt{k}\cos\left(\sqrt{k}L\right)=0$$

Ultimately, the solution for the temperature distribution can be expressed as

$$u(x,t)=\sum_{n=1}^{\infty}\left(A_n\cos\left(\sqrt{k}x\right)+B_n\sin\left(\sqrt{k}x\right)\right)e^{-\alpha kt}$$

3.2 Green's Function Method

The method of Green's functions is particularly useful for handling a broader range of boundary conditions, including non-homogeneous and complex boundary conditions, unlike the separation of variables method, which is typically applied in cases with symmetric regions and suitable boundary conditions.

For a given linear differential operator L defined on an appropriate domain Ω, the Green's function $G(x,y)$ is defined to satisfy the equation

$$LG(x,y)=\delta(x-y)$$

Given a source term $f(x)$, we consider the Laplace equation within the region Ω

$$\Delta u(x)=f(x),\quad x\in\Omega$$

with the boundary condition

$$u(x)=g(x),\quad x\in\partial\Omega$$

The first step is to construct the Green's function. Based on symmetry and physical intuition, we typically assume that the Green's function is symmetric, i.e., $G(x,y)=G(y,x)$. Moreover, the form of $G(x,y)$ can be derived based on the shape of the region and the boundary conditions.

3.2.1 Example Laplace's Equation in a Unit Circle

Consider the Laplace equation in a unit circle with the boundary condition $u=0$ on the boundary. In the unit circle $D=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2<1\}$, we examine the Laplace equation

$$\Delta u=0$$

with the boundary condition

$$u=0$$

To construct the Green's function $G(x,y)$, we require that for any point $\mathbf{x}=(x_1,y_1)\in D$ and a source point $\mathbf{y}=(x_2,y_2)\in D$

1. $\Delta G(\mathbf{x}, \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y})$ in D. 2. $G(\mathbf{x}, \mathbf{y}) = 0$ on ∂D.

For the fundamental solution of the planar Laplace equation, let $\mathbf{x} = (x_1, y_1)$ and $\mathbf{y} = (x_2, y_2)$ be two points in the plane (with the default Euclidean distance). The fundamental solution is given by

$$G_0(\mathbf{x}, \mathbf{y}) = -\frac{1}{2\pi} \ln |\mathbf{x} - \mathbf{y}|$$

To satisfy the zero boundary condition $G(\mathbf{x}, \mathbf{y})$ on the boundary, we can introduce a reflection point. Let $\mathbf{y}^* = \frac{\mathbf{y}}{|\mathbf{y}|^2}$ be the reflection of point $\mathbf{y}$ across the unit circle. Thus, we define the modified Green's function as

$$G(\mathbf{x}, \mathbf{y}) = -\frac{1}{2\pi} \left(\ln |\mathbf{x} - \mathbf{y}| - \ln |\mathbf{x} - \mathbf{y}^*| \right)$$

On the boundary ∂D, assuming $\mathbf{x}$ lies on the unit circle (i.e., $|\mathbf{x}| = 1$), we find that $|\mathbf{x} - \mathbf{y}^*| = |\mathbf{x}||\mathbf{y}^*| = 1$

$$G(\mathbf{x}, \mathbf{y}) = -\frac{1}{2\pi} \left(\ln |\mathbf{x} - \mathbf{y}| - \ln 1 \right) = 0$$

Thus, the constructed Green's function $G(\mathbf{x}, \mathbf{y})$ satisfies the boundary condition $G = 0$ on ∂D. Therefore, the Green's function for the Laplace equation within the unit circle, satisfying the boundary condition $u = 0$, is given by

$$G(\mathbf{x}, \mathbf{y}) = -\frac{1}{2\pi} \left(\ln |\mathbf{x} - \mathbf{y}| - \ln \left| \mathbf{x} - \frac{\mathbf{y}}{|\mathbf{y}|^2} \right| \right).$$

3.3 Variational Method

Energy can often be expressed in integral form, and the optimization problems involving functionals are known as variational problems. For the Laplace equation, the fundamental idea of the variational method is to construct a functional such that its extrema correspond to solutions of the Laplace equation.

The core of the variational method lies in the Euler-Lagrange equation. Consider a functional

$$J[u] = \int_\Omega F(x, u, \nabla u) \, \mathrm{d}x$$

We assume that the function u is fixed at the endpoints a and b. Let's introduce a small perturbation $u + \varepsilon\eta$ into J

$$J(u + \varepsilon\eta) = \int_a^b F(x, u + \varepsilon\eta, u' + \varepsilon\eta') \, \mathrm{d}x$$

By differentiating with respect to ε and then taking the limit as ε approaches zero, we obtain

$$\left. \frac{\mathrm{d}J}{\mathrm{d}\varepsilon} \right|_{\varepsilon=0} = \int_a^b \left(\frac{\partial F}{\partial u} \eta + \frac{\partial F}{\partial u'} \eta' \right) \mathrm{d}x$$

Using integration by parts on the term $\frac{\partial F}{\partial u'}\eta'$

$$\int_a^b \frac{\partial F}{\partial u'}\eta' \,\mathrm{d}x = \left[\frac{\partial F}{\partial u'}\eta\right]_a^b - \int_a^b \left(\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial u'}\right)\eta \,\mathrm{d}x$$

Since $\eta(a) = \eta(b) = 0$, the boundary term vanishes.

Substituting the above result into $\frac{\mathrm{d}J}{\mathrm{d}\varepsilon}$

$$\left.\frac{\mathrm{d}J}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} = \int_a^b \left(\frac{\partial F}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial u'}\right)\eta \,\mathrm{d}x$$

For this expression to be zero for all $\eta(x)$, we must have

$$\frac{\partial F}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial u'} = 0$$

Thus, we obtain the condition for the functional to attain an extremum.

The first step we need to take is to construct an energy functional such that when this functional reaches an extremum, it satisfies the Laplace equation. For the Laplace equation, the energy functional is typically given by the Dirichlet energy

$$J(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 \,\mathrm{d}x$$

Following the aforementioned procedure, we perform the variational analysis to find the function that maximizes or minimizes the functional, and then we apply the boundary conditions to obtain the solution. (This part is for understanding.)

References

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