Basic Properties and Theorems of L^pSpaces

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1 Introduction

In mathematical analysis, the function spaces we often discuss refer to collections of functions with certain specific properties, and the definition of function spaces depends on how the "distance" or "magnitude" between these functions is measured. In L^p spaces, this "magnitude" is defined through the integral of the p-th power.

In applied fields, many partial differential equations in physics and engineering require the use of L^p spaces to describe the existence and uniqueness of solutions, as well as certain classical solution theories relying on the properties of L^p spaces. *In short, it's very important!*

In this article, I will cover the basics and fundamental properties of L_p spaces. I recommend that readers first read the first article in the real analysis column to have a basic understanding and comprehension of the foundational concepts in real analysis and functional analysis before reading this article.

2 Some Aspects of Norms

2.1 Distance

The so-called "distance" is a concept that measures the "closeness" between any two points. For example, the Euclidean distance, which we have known since childhood, is defined as satisfying non-negativity, symmetry, the triangle inequality, and reflexivity to be considered a distance.

Let's take a classic example—the Minkowski distance. For two points $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ in n-dimensional Euclidean space, considering any real number $p \geq 1$, their Minkowski distance is defined as

$$d(\mathbf{x}, \mathbf{y}) = \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{1/p}$$

The Minkowski distance geometrically provides different shapes of level curves. Depending on the different values of p, the level curves exhibit different geometric shapes. For example, in two-dimensional space:

1. When $p = 1$, the level curve is a diamond shape (Manhattan geometry).
2. When $p = 2$, the level curve is a circle (Euclidean geometry).

3. When $p = \infty$, the level curve is a square (Chebyshev geometry).

2.2 p-Norm

Further, we understand the concept of **norm**. We know that a norm should possess positivity, homogeneity, and satisfy the triangle inequality. Sometimes, for the convenience of beginners, norms may be *informally* described as a concept of distance that is closer to our everyday language, *but this should not be confused in formal theoretical study.*

In **normed vector spaces**, we generally define the distance measure as the p-th power sum of its components followed by taking the p-th root, hence it is referred to as the p-norm. For example, when $p = 1$, it is called the Manhattan norm (L1, $\|\mathbf{x}\|_1 = \sum_{i=1}^{n} |x_i|$); when $p = 2$, it is the Euclidean norm (L2, $\|\mathbf{x}\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$).

When the space dimension is finite or countably infinite, for a vector $\mathbf{x}$ in $\mathbb{R}^n$ and a non-negative real number p, we define its p-norm as

$$\|\mathbf{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}$$

However, in more general function spaces or high-dimensional spaces, we cannot define norms using the method for finite or countable dimensional spaces. In such cases, the definition of the p-norm is given by Lebesgue integration.

Given a measure space $(X, \mathcal{M}, \mu)$ and a generalized real-valued function $f : X \to \overline{\mathbb{R}}$, if f is a measurable function on X with $p \in [0, \infty]$, then we denote $\|f\|_p \in [0, \infty]$ as the p-norm of f. When p is finite, it is defined as

$$\|f\|_p = \left(\int_X |f|^p \, \mathrm{d}\mu \right)^{\frac{1}{p}}$$

When $p = \infty$, it is called the uniform norm, defined as:

$$\|f\|_\infty = \inf\{M \in \mathbb{R}_{\geq 0} : |f| \leq M\}$$

In general, when $\mu(X) > 0$, we refer to $\|f\|_\infty$ as the essential infimum of f, meaning the supremum in the measure sense: the supremum of the function on the remaining set after ignoring sets of measure zero.

It is worth noting that although we often refer to p-norms, when $p < 1$, the p-norm is *not a valid norm by definition.* When $p = 0$, the homogeneity of the norm is not satisfied, as $\|kx\|_0 \neq |k| \cdot \|x\|_0$ does not necessarily hold. When $0 < p < 1$, the triangle inequality is not satisfied. Consider two non-zero vectors $x = 1$ and $y = -1$, we have:

$$\|x + y\|_p = (|1|^p + |1|^p)^{\frac{1}{p}} = 2^{\frac{1}{p}}$$

$$\|x\|_p + \|y\|_p = 1 + 1 = 2$$

Clearly, in this case, $2^{1/p} > 2$, hence the triangle inequality is not satisfied.

Definition 2.1. The p-norm of a vector $\mathbf{v}$ when $p = 0$ is usually defined as the number of non-zero elements in a vector:

$$\|\mathbf{v}\|_0 = \#\{i \mid x_i \neq 0\}$$

3 L^p Spaces and Their Properties

3.1 L^p Spaces

The L^p space is a special type of normed linear space, consisting of the set of all functions that have a finite p-norm over a given domain. The L^p space is a collection of measurable functions defined under Lebesgue measure, and it is sometimes referred to as Lebesgue space. The L^p space is also a typical Banach space; in particular, when $p = 2$, it is the Hilbert space $L^2(S^1)$, commonly used in Fourier analysis.

In terms of definition, L^p spaces can be real or complex. For convenience, let the field $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$. Given a measure space $(X, \mathcal{M}, \mu)$, for $1 \leq p < \infty$, the L^p space is defined as follows:

$$L^p(X, \mu) = \left\{ f : X \to \mathbb{K} \;\middle|\; f \text{ measurable, } \|f\|_p < \infty \right\}$$

The term L^p in this article will be used to collectively refer to a class of spaces that satisfy this definition.

The L^p space is a vector space, typically defined through the addition of functions and scalar multiplication with functions. For all $f, g \in L^p(X),\ \lambda \in \mathbb{K}$, we have: + $(f + g)(x) = f(x) + g(x)$ + $\lambda f(x) = (\lambda f)(x)$

In practice regarding the Lebesgue measure, we often use $L^p(\mathbb{R}^n, m)$, which is also one of the most typical and common examples in real analysis. Sometimes we further restrict the space to a specific interval, such as $L^p([a, b])$.

3.2 Related Inequalities

3.2.1 Hölder's Inequality

Recall the Cauchy-Schwarz inequality: for two vectors $\mathbf{v}$ and $\mathbf{u}$ in an inner product space V over the real or complex numbers (i.e., $\mathbb{K}$), we have

$$\|\langle \mathbf{v}, \mathbf{u} \rangle\| \leq \|\mathbf{v}\| \cdot \|\mathbf{u}\|.$$

Definition 3.1 (Hölder's Inequality)**.** Hölder's inequality is a generalization of the Cauchy-Schwarz inequality that we commonly use. Consider a measure space (X, μ), with real numbers $p, q \geq 1$, and functions $f, g \in L^p(X)$ satisfying

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Hölder's inequality is given by

$$\int_X |f(x)g(x)|\, d\mu \leq \left(\int_X |f|^p\, d\mu \right)^{\frac{1}{p}} \cdot \left(\int_X |g|^q\, d\mu \right)^{\frac{1}{q}}. \tag{1}$$

Reduction to Cauchy-Schwarz Inequality. It is easy to see that when we apply the Cauchy-Schwarz inequality to functions in L^2 space, by taking $p = q = 2$, Hölder's inequality reduces to the Cauchy-Schwarz inequality. ■

Proof. Hmm... looking at the definition, it seems familiar! You might be reminded of Young's inequality, which will make our proof quite straightforward.

Lemma 3.2 (Young's Inequality)**.** *Consider positive real numbers a, b, p, q, with $\frac{1}{p} + \frac{1}{q} = 1$. Then we have*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Equality holds if and only if $a^p = b^q$.

Consider the measure space (X, μ), with functions $f \in L^p(X)$ and $g \in L^q(X)$, where $\frac{1}{p} + \frac{1}{q} = 1$ and both $\|f\|_p$ and $\|g\|_q$ are non-zero. Applying Young's inequality, we have

$$\begin{aligned} \int_X |f(x)g(x)| \, \mathrm{d}\mu &\leq \left(\int_X |f|^p \, \mathrm{d}\mu\right)^{\frac{1}{p}} \cdot \left(\int_X |g|^q \, \mathrm{d}\mu\right)^{\frac{1}{q}} \\ \frac{1}{\|f\|_p \|g\|_q} \int_X |f(x)g(x)| \, \mathrm{d}\mu &\leq 1. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} \frac{|f|}{\|f\|_p} \cdot \frac{|g|}{\|g\|_q} &\leq \frac{1}{p}\left(\frac{|f|}{\|f\|_p}\right)^p + \frac{1}{q}\left(\frac{|g|}{\|g\|_q}\right)^q \\ \int_X \frac{|f|}{\|f\|_p} \cdot \frac{|g|}{\|g\|_q} \, \mathrm{d}\mu &\leq \frac{1}{p} + \frac{1}{q} = 1 \\ \int_X |fg| &\leq \left(\int_X |f|^p \, \mathrm{d}\mu\right)^{\frac{1}{p}} \cdot \left(\int_X |g|^q \, \mathrm{d}\mu\right)^{\frac{1}{q}}. \end{aligned}$$

Proof completed. ■

Conditions for Equality

The conditions for equality in Hölder's inequality have two cases:

1. When $p = \infty$ or $q = \infty$, the functions f and g have almost everywhere the same support.
2. When $p > 1$ and $q < \infty$, there exists a constant $\lambda \geq 0$ such that almost everywhere:

$$|f|^p = \lambda |g|^q$$

3.2.2 Minkowski Inequality

The Minkowski inequality has several equivalent forms.

First, in norm form, consider finite positive real numbers $1 < p < \infty$ and functions $f \in L^p(X), g \in L^q(X)$. We have:

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p$$

When $p < 1$, we know the triangle inequality does not hold, so we consider two positive measure sets $A, B \subset X$ where $A \cap B = \emptyset$, and f, g are indicator functions, yielding $\|f + g\| \geq \|f\|_p + \|g\|_p$.

It is easy to see that when $p = 2$, the Minkowski inequality reduces to the triangle inequality in Euclidean space.

Next is the integral form. For real numbers $p > 1$, if f, g are integrable on $[a, b]$, we have:

$$\left(\int_a^b (|f+g|)^p\right)^{\dfrac{1}{p}} \leq \left(\int_a^b |f|^p\right)^{\dfrac{1}{p}} + \left(\int_a^b |g|^p\right)^{\dfrac{1}{p}}$$

Similarly, when $0 \leq p \leq 1$ and both f, g are non-negative, the inequality is reversed.

It can also be expressed in a discrete form:

$$\left(\sum_{k=1}^{n} |x_k + y_k|^p\right)^{\frac{1}{p}} \leq \left(\sum_{k=1}^{n} |x_k|^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{n} |y_k|^p\right)^{\frac{1}{p}}$$

Proof. One of the most common methods to prove the Minkowski inequality uses Hölder's inequality. If $f + g = 0$, or if $p = 1$ or $p = \infty$, the inequality clearly holds. Otherwise, we first expand some terms:

$$\begin{aligned}
(\|f+g\|_p)^p &= \sum_{k=1}^{n} |f_k + g_k|^p \\
&= \sum_{k=1}^{n} |f_k + g_k| \cdot |f_k + g_k|^{p-1} \\
&\leq \sum_{k=1}^{n} (|f_k| + |g_k|) \cdot (|f_k + g_k|)^{p-1} \\
&= \sum_{k=1}^{n} |f_k| \, |f_k + g_k|^{p-1} + \sum_{k=1}^{n} |g_k| \, |f_k + g_k|^{p-1} \\
&\leq \left(\sum_{k=1}^{n} |f_k|^p\right)^{\frac{1}{p}} \cdot \left(\sum_{k=1}^{n} \left(|f_k + g_k|^{p-1}\right)^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \\
&\quad + \left(\sum_{k=1}^{n} |g_k|^p\right)^{\frac{1}{p}} \cdot \left(\sum_{k=1}^{n} \left(|f_k + g_k|^{p-1}\right)^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \\
&= (\|f\|_p + \|g\|_p) \cdot (\|f+g\|_p)^{p-1}
\end{aligned}$$

The expression is quite long and not very elegant, but we can simply cancel the common factor:

$$\begin{aligned}
&\|f+g\|_p^p \leq (\|f\|_p + \|g\|_p) \cdot (\|f+g\|_p)^{p-1} \\
&\|f+g\|_p \leq \|f\|_p + \|g\|_p
\end{aligned}$$

Proof completed. ■

3.3 Basic Properties

- The L^p space is a linear space. From the inequality $|f+g|^p \leq 2^{p-1}(|f|^p + |g|^p)$, it follows that for functions $f, g \in L^p$ and any real numbers a, b, the linear combination $af + bg$ is also in L^p.
- When $1 \leq p < \infty$, the space $L^p(X)$ is separable.
- When $1 \leq p < \infty$, for a topological space X and its Radon measure μ, the following sets are dense in $L^p(X)$:
  - Simple integrable functions, which take a finite number of values, with each function value corresponding to a measurable set of finite measure.
  - Bounded p-integrable functions, which are bounded over their entire domain and belong to L^p, forming a dense subset in L^p.
  - The set of continuous functions with compact support.

3.4 Embedding Properties

For $1 \leq p < q < \infty$, the space L^q is embedded in L^p, i.e., $L^q \subset L^p$.

This embedding relationship is **continuous**. Consider a sequence $\{f_n\} \in L^q$ that converges in the L^q norm and converges to g in the L^p norm. By extracting a subsequence from $\{f_n\} \in L^q$ that converges almost everywhere to f, it still converges to g in the L^p norm. From this subsequence, we can extract another subsequence that also converges almost everywhere to g. Due to the consistency of limits, $f = g$ holds almost everywhere. Since both L^q and L^p are Banach spaces, this embedding map is continuous.

Definition 3.3 (Closed Graph Theorem)**.** Let X and Y be Banach spaces, and let $T : X \to Y$ be a linear operator. The graph of T, denoted by Γ_T, is defined as:

$$\Gamma_T = \{(x, T(x)) \in X \times Y \mid x \in \mathrm{dom}(T)\}.$$

The operator T is said to be a **closed operator** if its graph Γ_T is closed in $X \times Y$. According to the Closed Graph Theorem, if T is closed and its domain $\mathrm{dom}(T)$ is a closed set in X, then T is continuous.

3.5 Dual Space

Given an L^p space, we can define its dual space $L^p(\Omega)^*$. The dual space consists of all continuous linear functionals that act on L^p.

When $1 \leq p \leq \infty$, consider q satisfying $\frac{1}{p} + \frac{1}{q} = 1$ (the conjugate exponent). The dual space of L^p is given by L^q.

When $1 < p < \infty$ or $p = 1$ and X is a σ-finite measure space, the dual space $(L^p)^*$ is isometrically isomorphic to L^q. That is, given a bounded linear functional in L^p:

$$\phi(f) = \int_X f(x)g(x)\mathrm{d}\mu(x)$$

it can be uniquely represented as $g \in L^p$ while maintaining the norm:

$$\|\phi\|_{(L^p)^*} = \|g\|_q$$

When $p = 1$, according to the Riesz representation theorem, the dual space of L^1 is isomorphic to L^∞.

More importantly, in the case when $p = \infty$, we know that $L^\infty = (L^1)^*$ and that the closed unit ball is compact in the weak* topology $\sigma(L^\infty, L^1)$.

3.6 Completeness (Riesz/Fischer Theorem)

Theorem 3.4 (Riesz/Fischer Theorem)**.** *When $1 \leq p < \infty$, the space L^p is a complete space.*

Proof. Consider $1 \leq p < \infty$, and let $\{f_n\}$ be a Cauchy sequence in L^p. We can choose a subsequence $\{f_{n_k}\}$ such that

$$\|f_{n_{k+1}} - f_{n_k}\|_{L^p} \leq \frac{1}{2^k}.$$

We will show that $\{f_{n_k}\}$ converges in L^p. Define $g_k = f_{n_{k+1}} - f_{n_k}$. Then, it follows that

$$\sum_{k=1}^{\infty} \|g_k\|_{L^p} < \infty.$$

Thus, the series

$$f_{n_1} + \sum_{k=1}^{\infty} g_k$$

converges almost everywhere in L^p to a function $f \in L^p$. Therefore, the limit of this subsequence exists:

$$\lim_{k\to\infty} f_{n_k} = \lim_{k\to\infty} \left(f_{n_1} + \sum_{i=1}^{k-1} g_i \right) = f_{n_1} + \sum_{k=1}^{\infty} g_k = f.$$

Since every Cauchy sequence converges, we conclude that $L^p(X)$ is complete for $1 \leq p < \infty$.

Now, consider the case $p = \infty$. If $\{f_n\}$ is a Cauchy sequence in L^∞, then for each $m \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that for all $k, j \geq n$,

$$|f_k - f_j| < \frac{1}{m}.$$

Define a measurable function $f : X \to \mathbb{R}$ that is almost everywhere uniquely determined by

$$f = \lim_{j\to\infty} f_j, \quad x \in N^c.$$

As $j \to \infty$, for each $m \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that for all $k \geq n$,

$$|f_k - f| \leq \frac{1}{m}.$$

Thus, f is essentially bounded, and as $k \to \infty$, $f_k \to f$ in L^∞.

Proof completed. ■